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Laminate Theory

Book Geoff Eckold, Chapter 3, pp 65-99

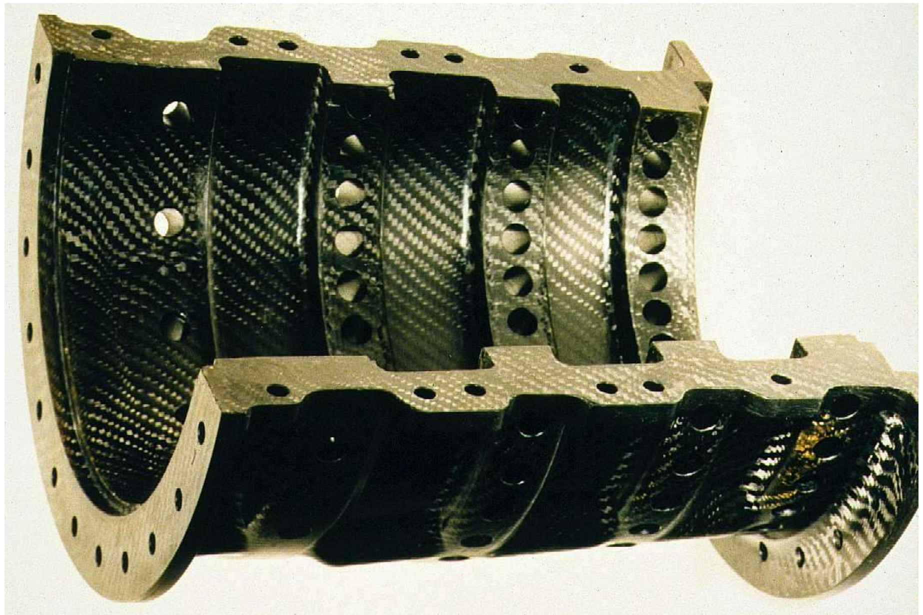
Designing rules for components made from polymer composites

Unlike to homogeneous materials, there are two design levels to consider:

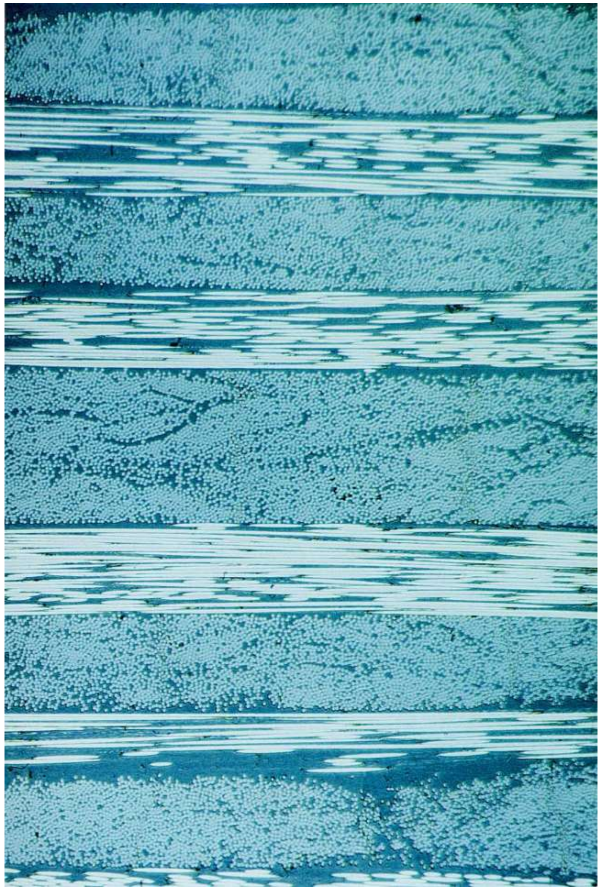
- **Designing the outer form of the component**
- **Designing the internal structure of the laminate**

Example of a component in Mechanical Engineering

- **Outer form**



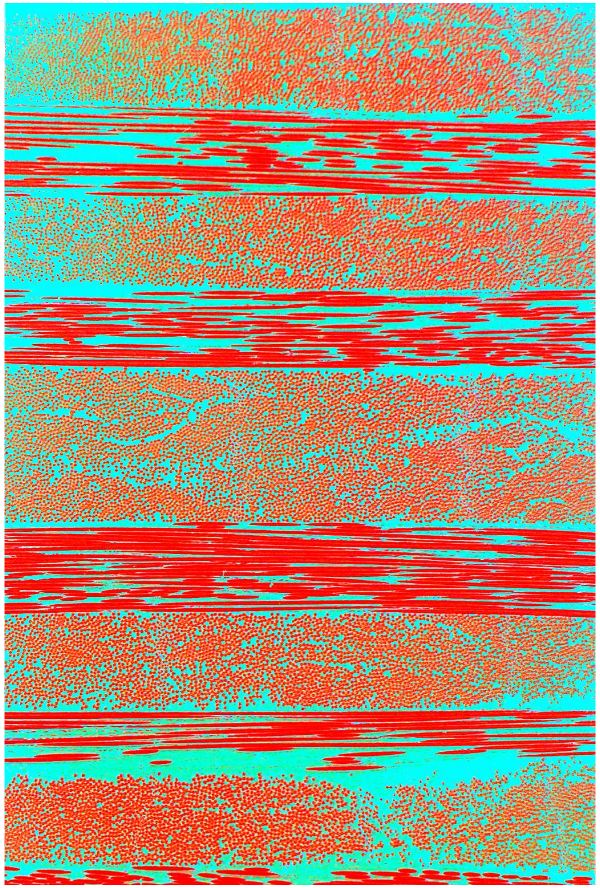
- **Internal laminate structure**



Civil Engineering

- **Outer form**

- **Internal laminate structure**

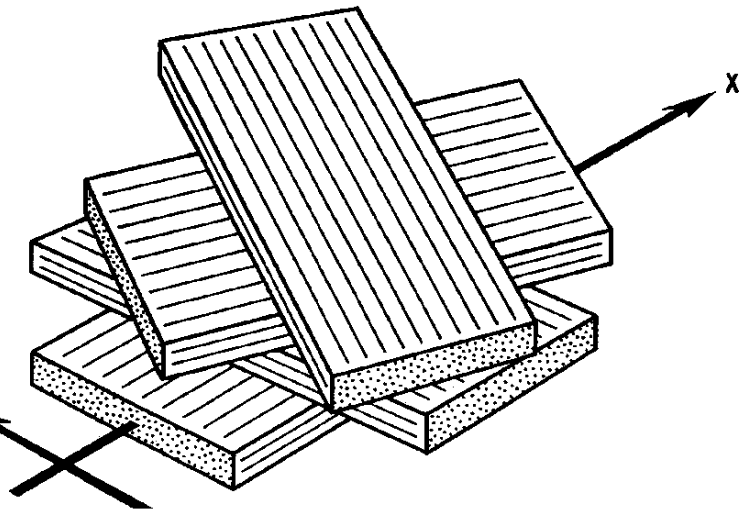


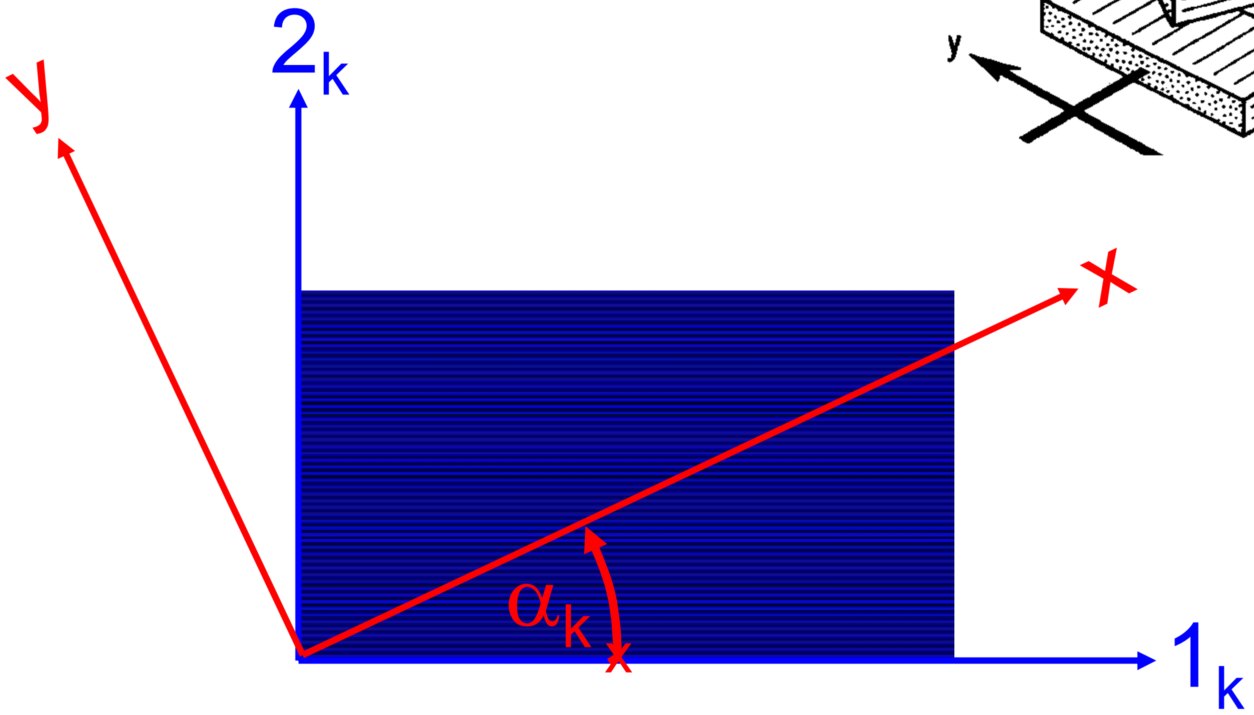
Following aspects have to be considered for the designing of internal laminate structure:

- **Fiber types**
- **Matrix**
- **Type of reinforcement material**
- **Laminate lay up**

Arrangement of Laminas (Layers) in a Laminate

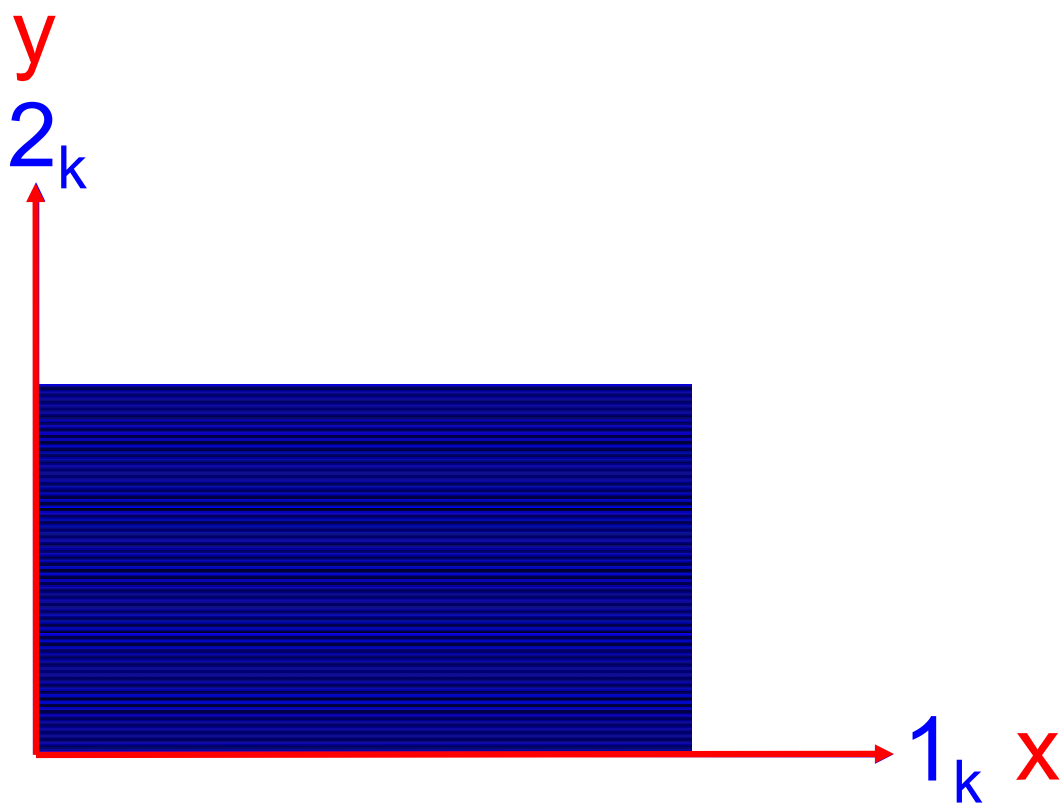
Lamina Angle α_k





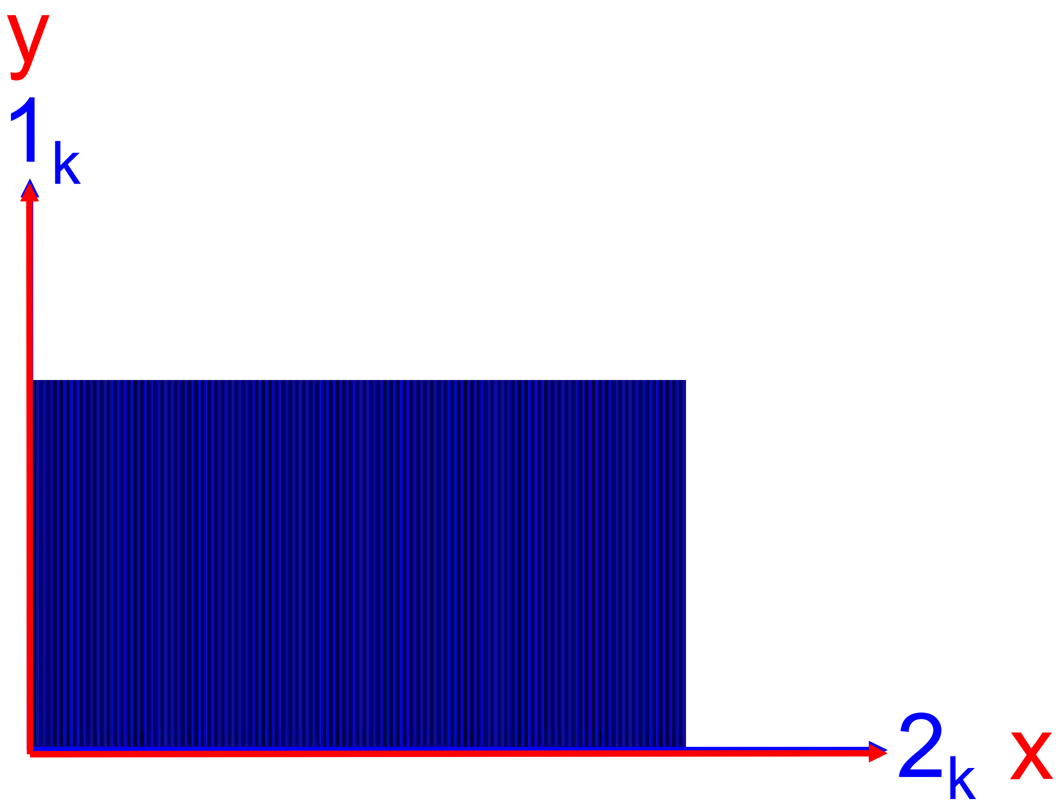
Arrangement of Laminas (Layers) in a Laminate

Lamina Angle : **0°- orientation ≡ fibers are parallel to x-axis**



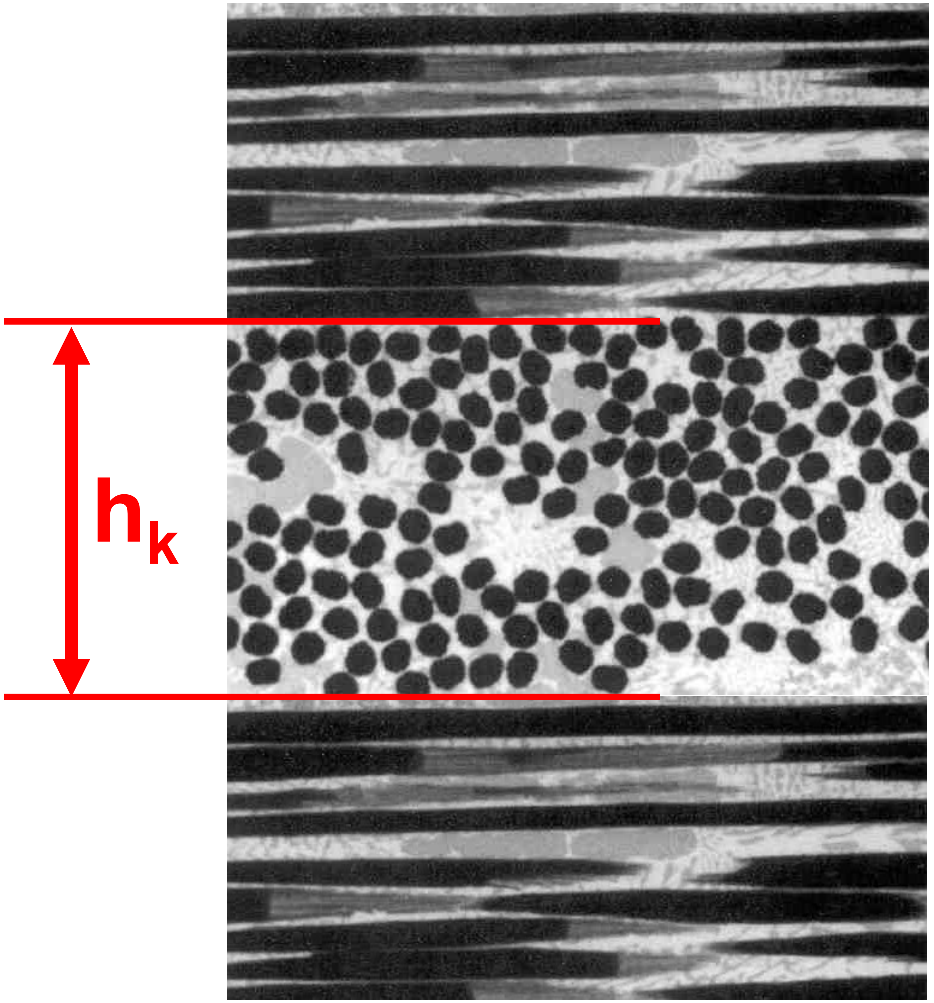
Arrangement of Laminas (Layers) in a Laminate

Lamina Angle : 90°- orientation ≡ fibers are parallel to y-axis



Arrangement of Laminas (Layers) in a Laminate

Lamina Thickness h_k



Defining a Laminate

- ***Symmetry:*** each layer is exactly mirrored about the geometric mid-plane in terms of its properties, thickness and orientation.
- ***Non-symmetric:*** laminates usually exhibit coupling between direct stress and curvature. For example, a 2-ply 0/90 cross-ply laminate will bend when in-plane tension is applied.
- ***Balanced:*** characterized by ±θ pairs of layers
- ***Note:***
  - ***Coupling*** is only removed entirely if the laminate is both balanced and symmetric about the centre line.
  - Remember that ***Balancing*** only applies if the laminate has off-axis plies.

Laminate Examples

0/90/90/0	*Symmetric*
0/90/0	*Symmetric*
0/90/0/90	*Non-symmetric - tension/bending coupling*
0/90/45/90/0	*Symmetric, Not Balanced - tension/shear coupling*
0/90/45/-45/-45/45/90/0	*Balanced and Symmetric*

Laminate Notation

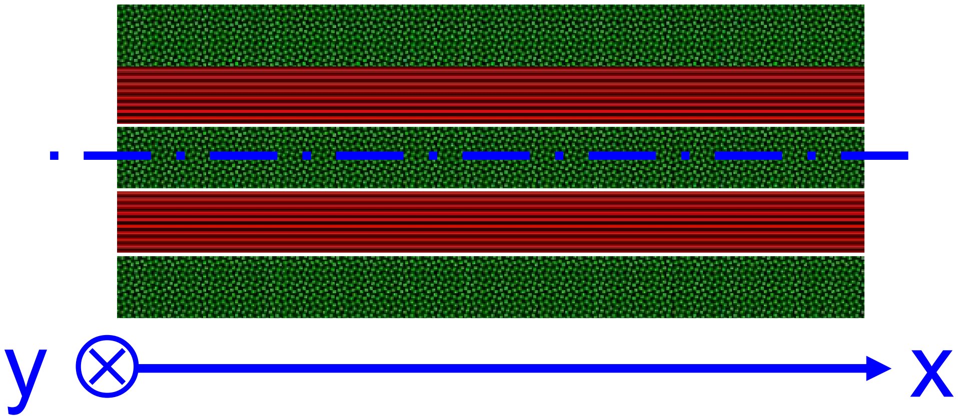
Brackets **[….]**	$[0, 90]_S$ = 0 / 90 / 90 / 0 *Square brackets denote a sequence of ply orientations.*
Subscript s **$[….]_s$**	$[0, 90]_S$ = 0 / 90 / 90 / 0 *The subscript s indicates symmetry*
Repeated plies θ_n	$[0_3, 90_2]_S$ = 0 / 0 / 0 / 90 / 90 / 90 / 90 / 0 / 0 / 0 $[0, (+45,-45)_2]_S$ = 0 / +45 / -45 / +45 / -45 / -45 / +45/ -45 / +45 / 0 *Repeated plies or ply sequences are demoted by a numerical subscript or the variable n*
Subscript T **$[….]_T$**	$[0/90/45/90/0]_T$=0 / 90 / 45 / 90 / 0 *Not all practical laminates are symmetrical, so the subscript 'T' is sometimes used to denote the total laminate lay up.*

Example A for arrangement of Laminas

Symmetric Crossply, consisting of three 90°-laminas and two 0°-laminas

Symmetry

(90° / 0° / 90° / 0° / 90°)

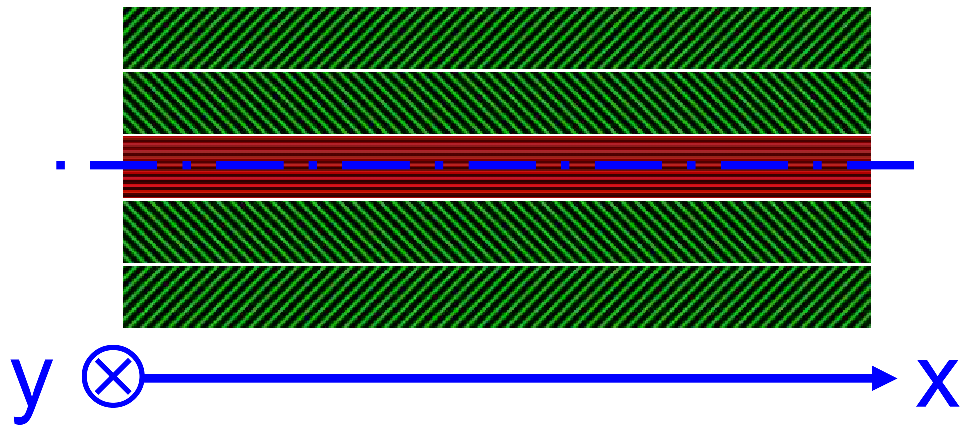


Example B for arrangement of Laminas

Balanced Angleply consisting of four +/-45°-laminas und one 0°-lamina.

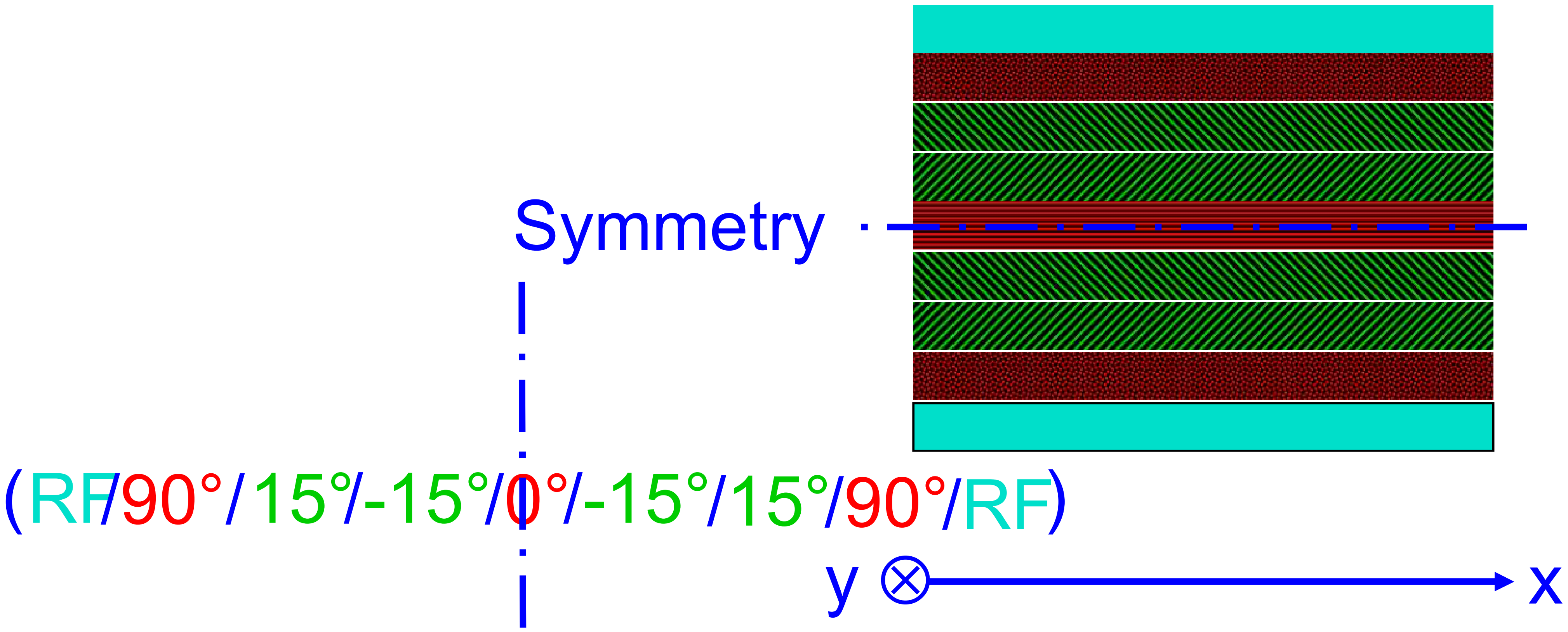
Symmetry

(45°/ -45°/ 0°/ -45°/ 45°)



Example C for arrangement of Laminas

Balanced Angleply consisting of Random Fiber (RF) laminas

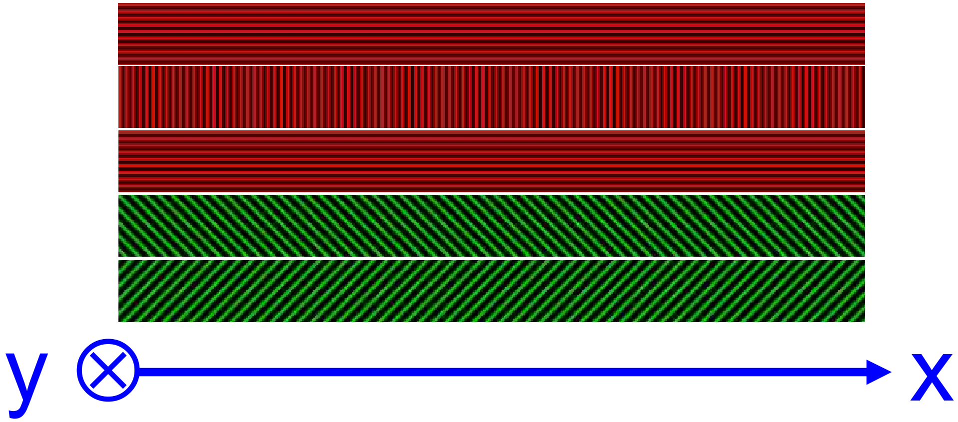


Example D for arrangement of Laminas

Balanced Angleply consisting of five laminas

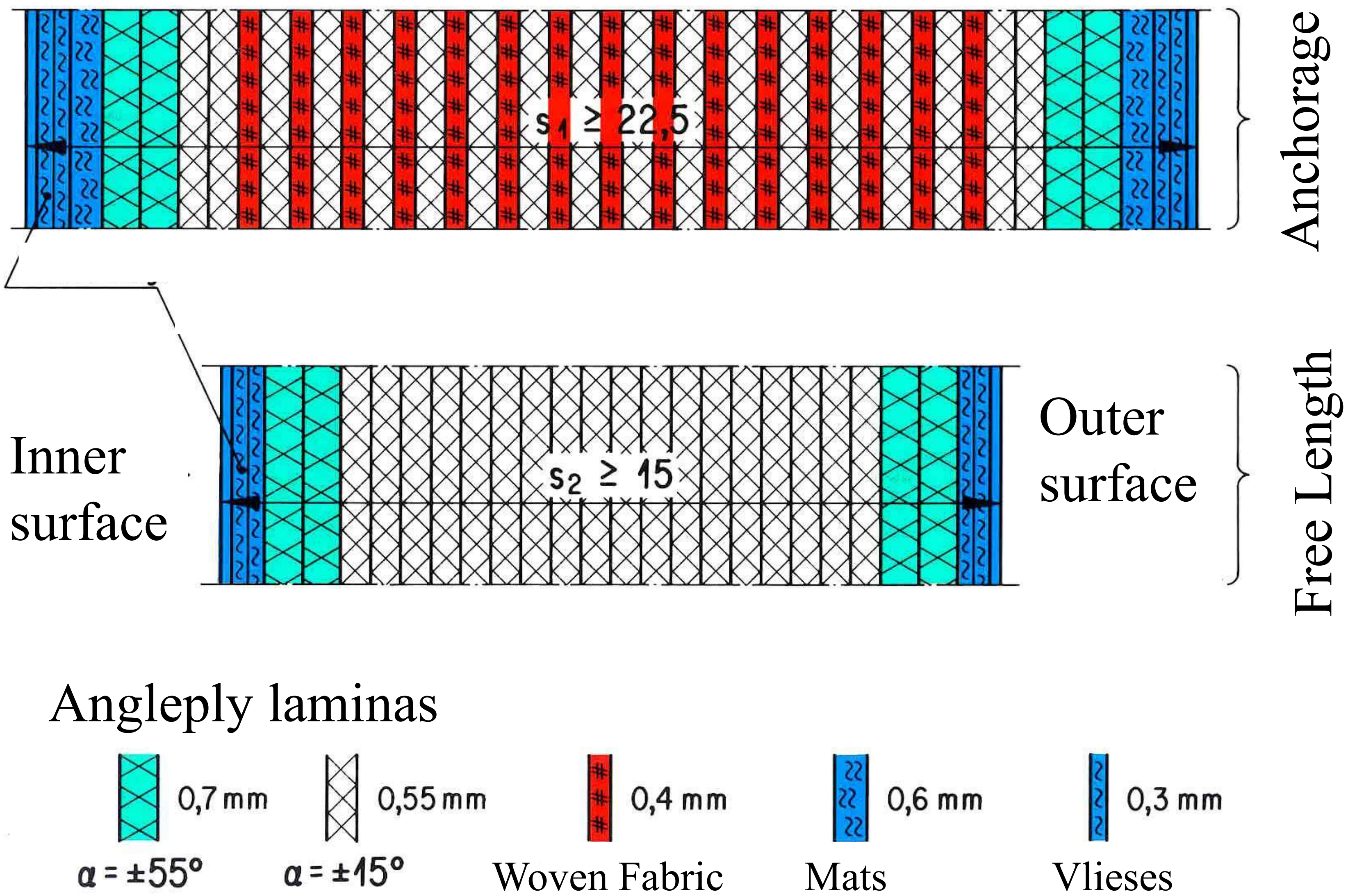
nonsymmetric!

(45°/ -45° / 0° / 90° / 0°)



Transmitter masts of Cima di Dentro

Laminate structure of the transmitter mast Cima di Dentro



Quiz: Laminate Structure

(- 45°/45°/90°/0°/0°/ 90°/ 45°/- 45°)

A) Balanced crossply

B) Unbalanced angleply

C) Unbalanced crossply

D) Balanced angleply

Answer to Quiz: Laminate Structure

„balanced“

(- 45°/45°/90°/0°/0°/ 90°/ 45°/- 45°)

A) Balanced crossply

B) Unbalanced angleply

C) Unbalanced crossply

D) balanced angleply

Quiz: Laminate Structure

(90°/90°/0°/ 90°/0°/0°/90°/90°)

A) Balanced crossply

B) Unbalanced angleply

C) Non-symmetric crossply

D) Balanced angleply

Answer to Quiz: Laminate Structure

(90°/90°/0°/ 90°/0°/0°/90°/90°)

„Non-Symmetric“

A) **Balanced crossply**

B) **Unbalanced angleply**

C) **Non-symmetric crossply**

D) **Balanced angleply**

Quiz: Laminate Structure

(90°/90°/0°/ 90°/0°/90°/90°)

A) Symmetric crossply

B) Unbalanced angleply

C) Unbalanced crossply

D) Balanced angleply

Answer to Quiz: Laminate Structure

„Symmetric“

(90°/90°/0°/ 90°/0°/90°/90°)

A) Symmetric crossply

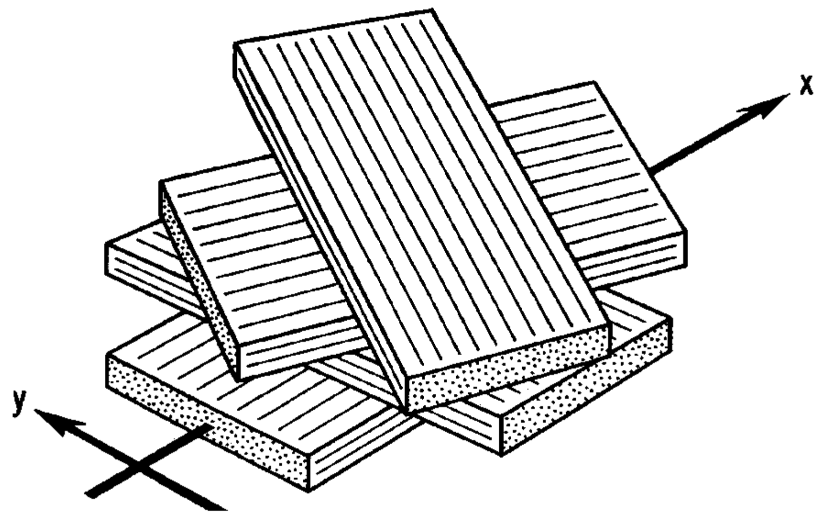
B) Unbalanced angleply

C) Unbalanced crossply

D) Balanced angleply

Quiz: Laminate Structure

(90°/0°/45°/-45°)



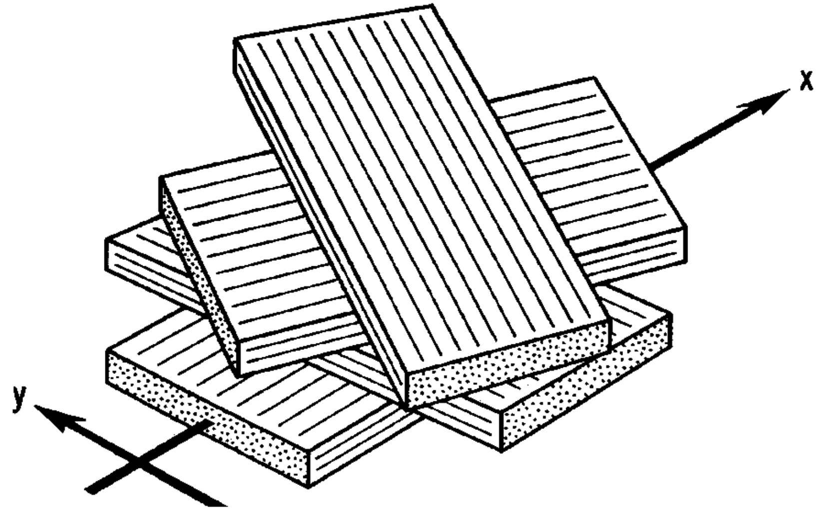
A) Balanced crossply

B) Unbalanced angleply

C) Unbalanced crossply

D) Balanced angleply

Answer to Quiz: Laminate Structure



(90°/0°/45°/-45°)

A) Balanced crossply

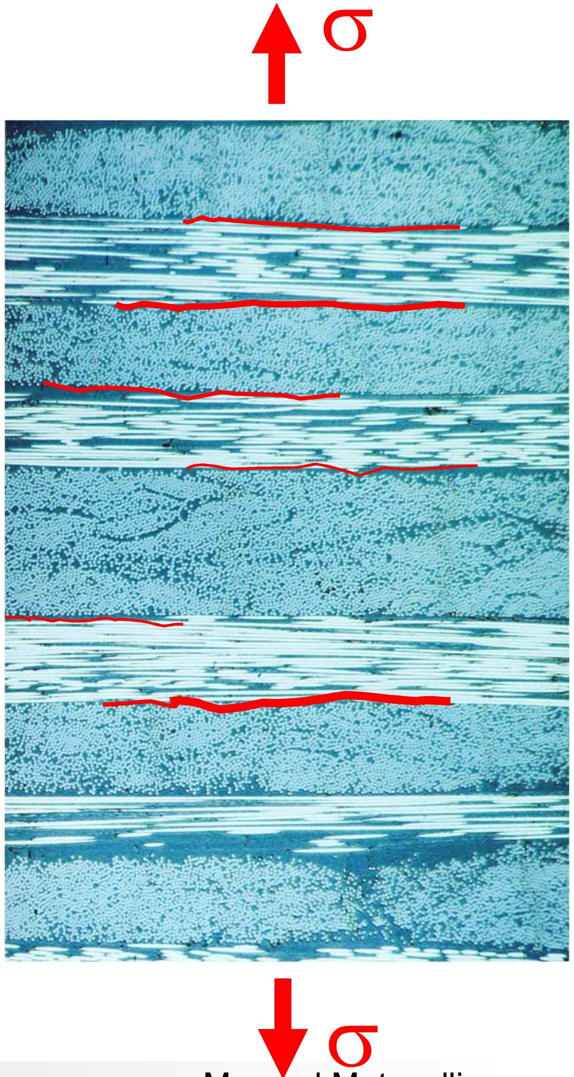
B) Unbalanced angleply

C) Unbalanced crossply

D) Balanced angleply

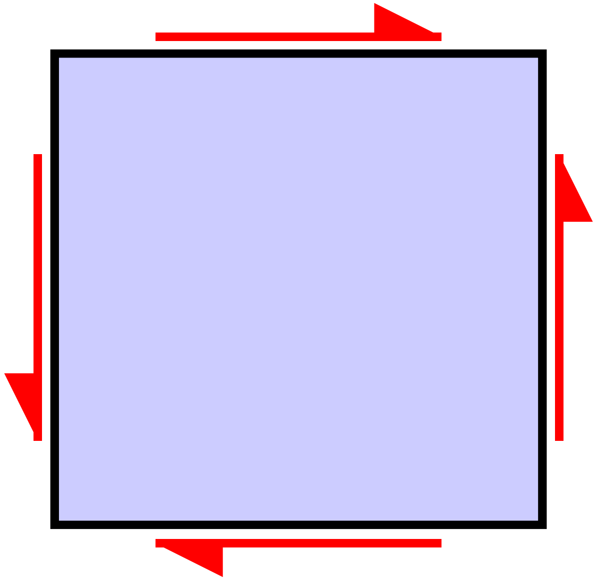
Fibers are usually in the laminate plane

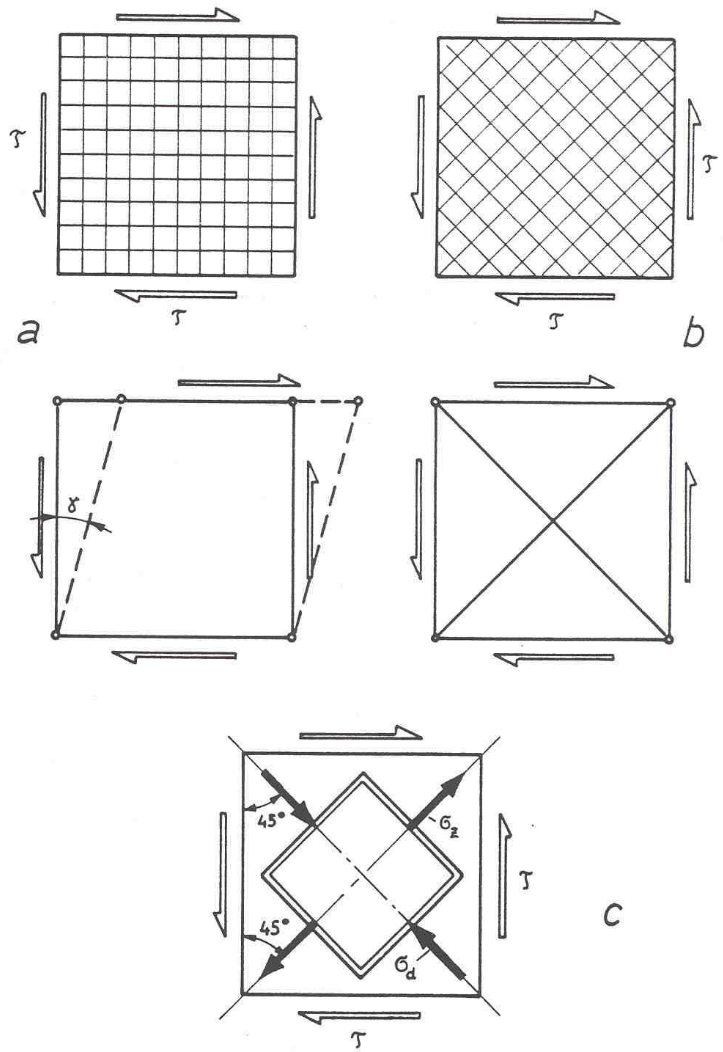
Stresses perpendicular to the laminate plane are carried by matrix and fiber/matrix interface. Therefore, such weaknesses have to be avoided!



Crossply laminate is weak against τ

Framework Model

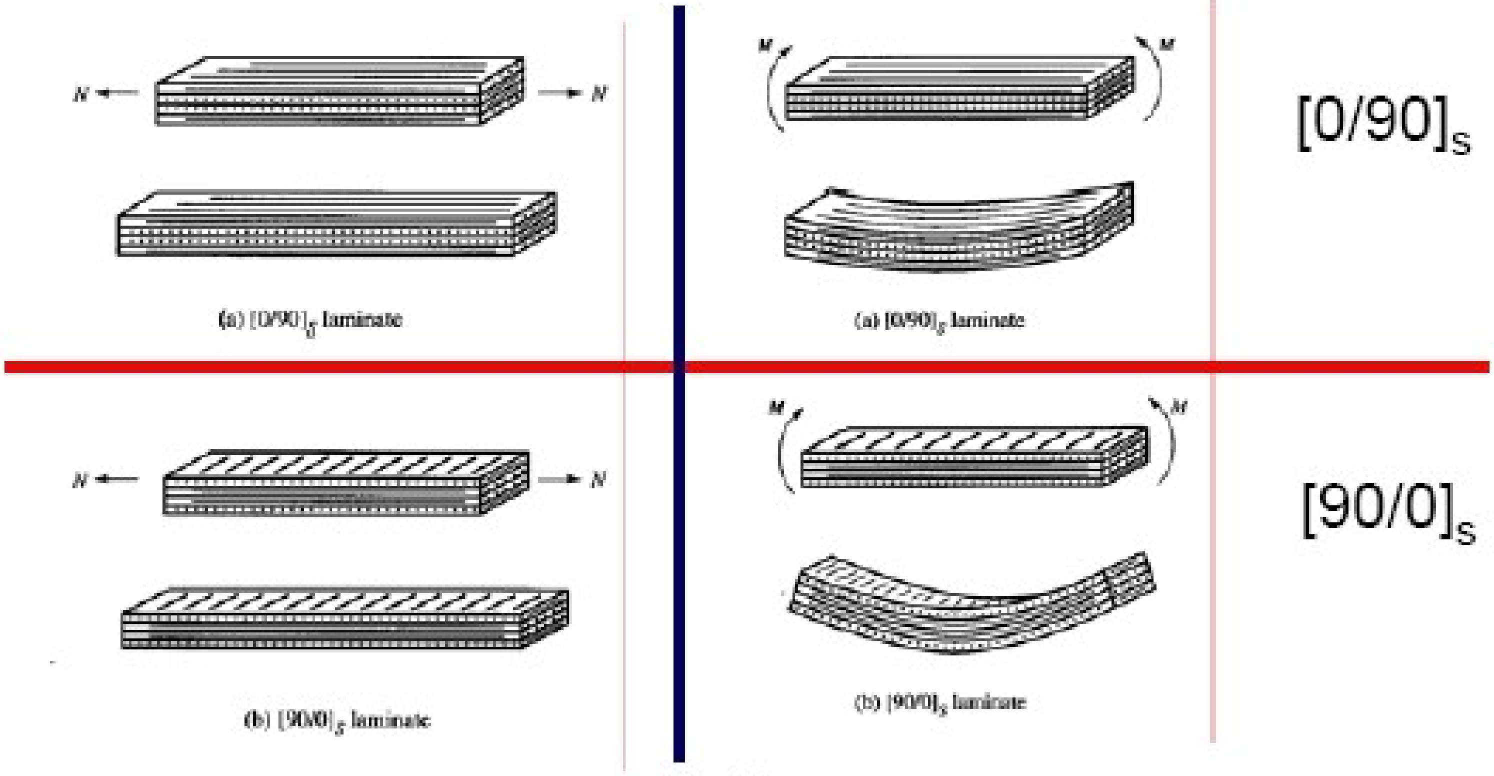




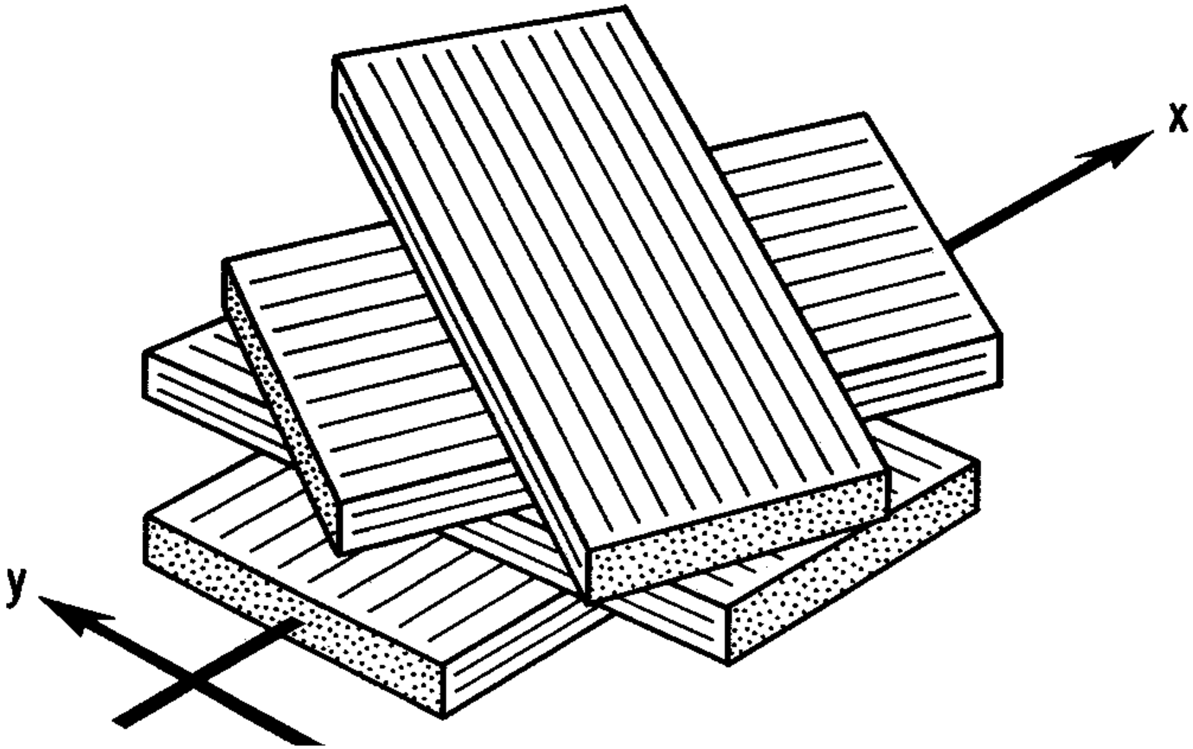
Classical Laminate Theory

Classical Lamination Theory

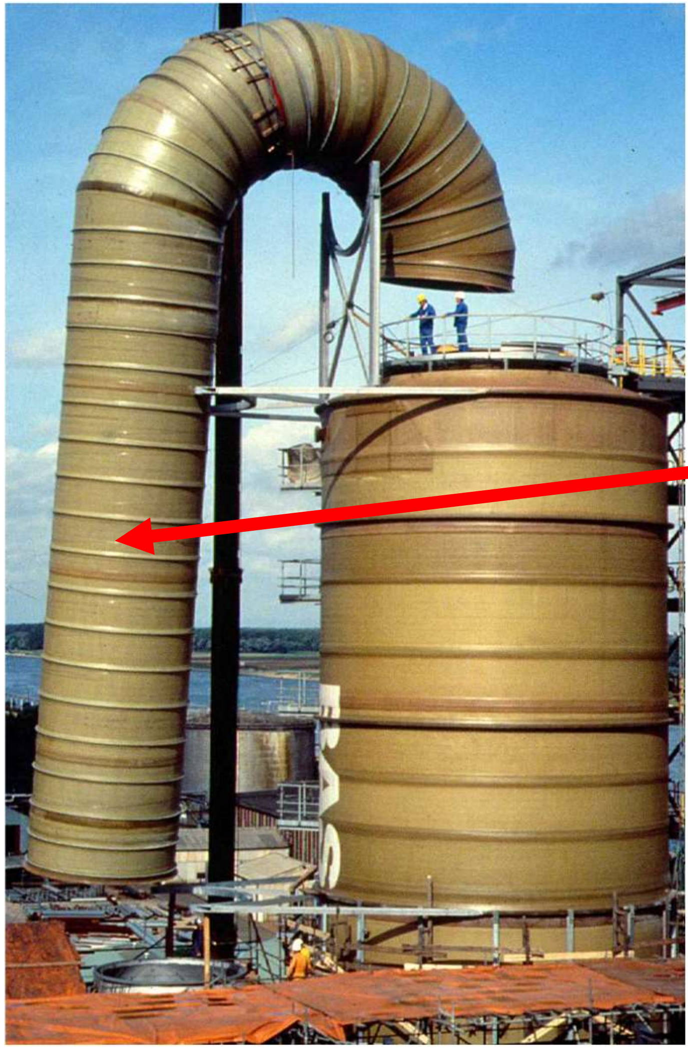
How do we treat a multilayered laminate with varied fiber orientation, stacking sequence and ply level material properties?

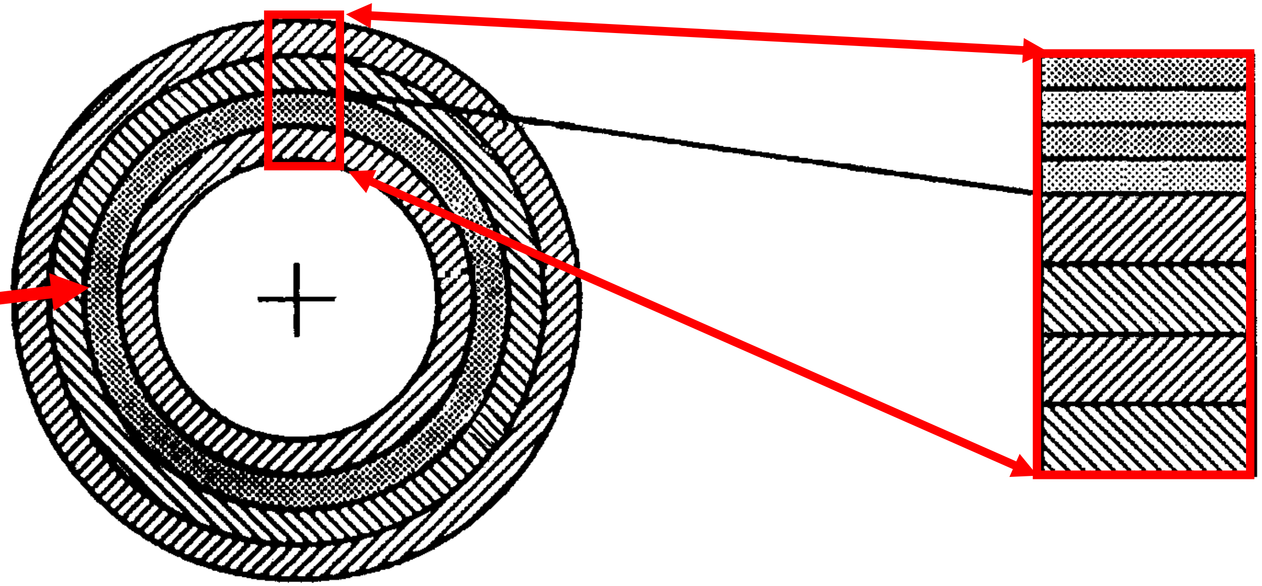


Classical Laminate Theory

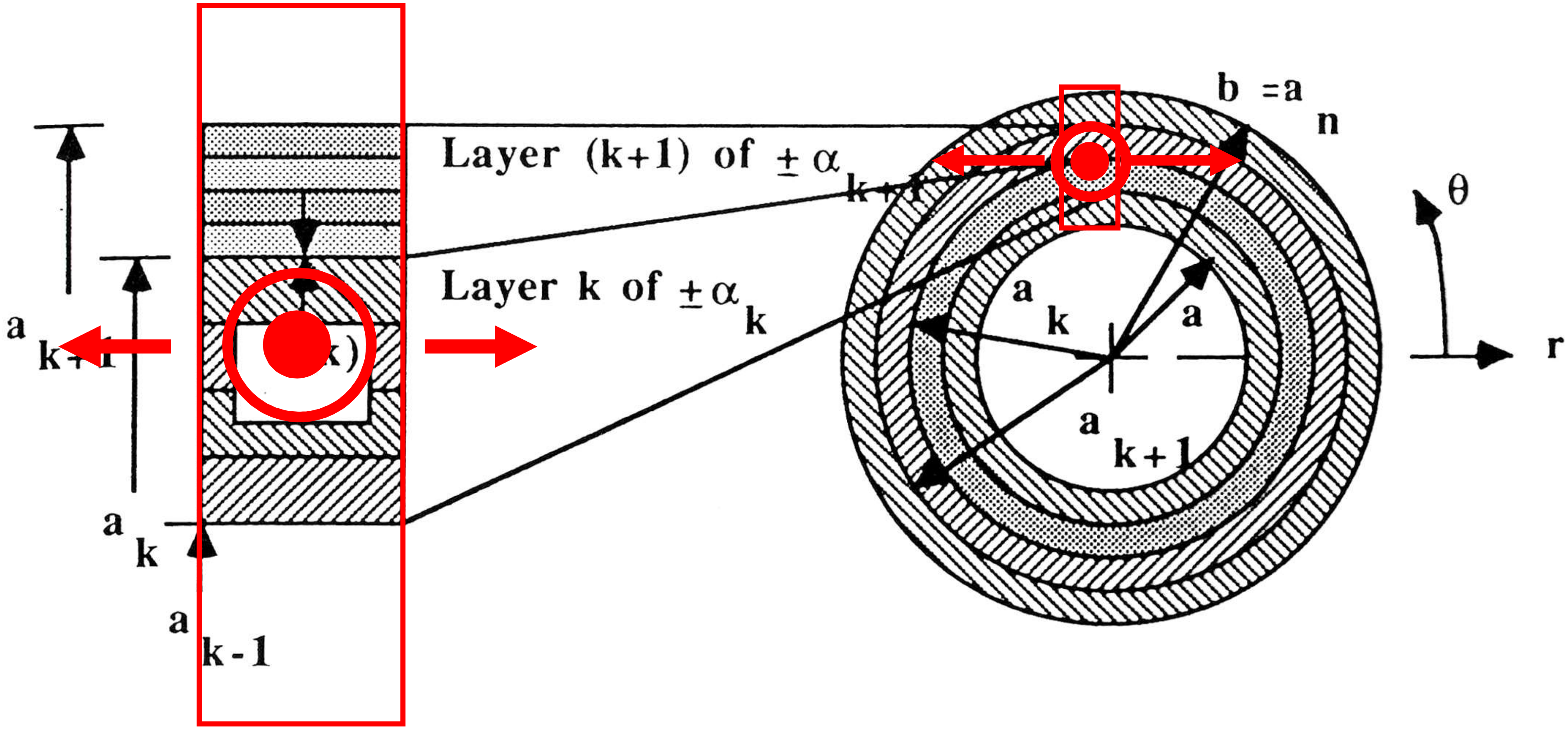


„Small“ Element

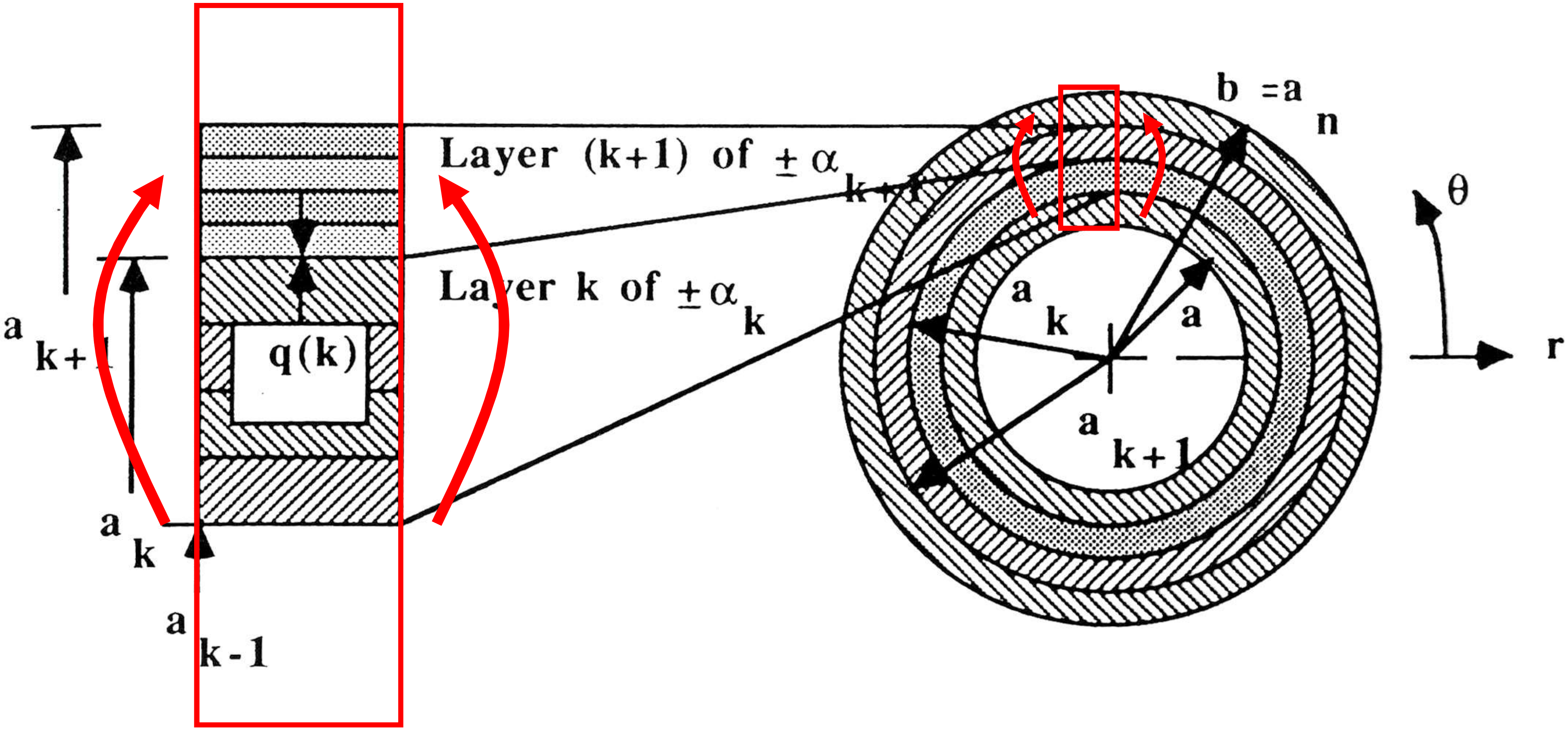




„Small“ Element



„Small“ Element



Assumptions

- **Perfect bond between single laminas**
- **Hook's law is valid**

Assumptions and Definitions

The elasticity constants of a laminate are calculated from the elasticity constants of the laminas applying equilibrium equations, compatibility conditions and Hook‘s law

Assumptions and Definitions

The classical membrane and plate theory are the foundation for the stress and deformation analysis

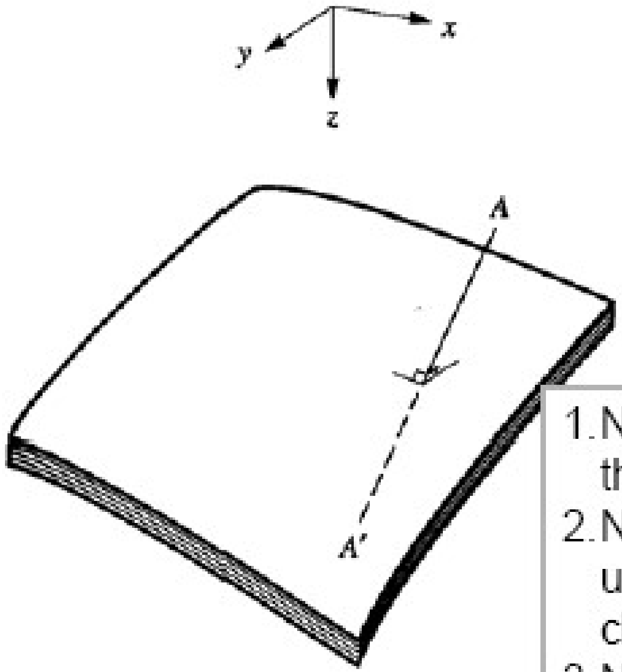
Basic Assumptions – Classical Lamination Theory

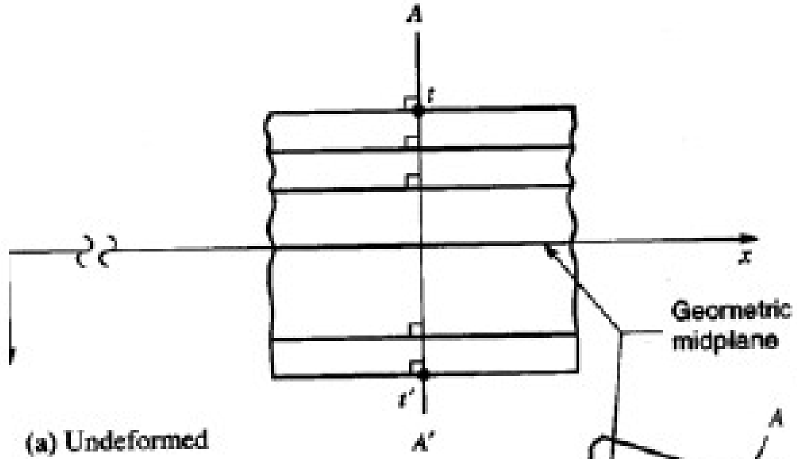
- Similar to the **Euler-Bernoulli beam** and the **Plate** theory, the classical lamination theory is only valid for thin laminates
  - Span > 10× thickness h
  - Small displacement w in the transverse direction (w << h)
- Shares the same classical plate theory assumptions – **Kirchhoff Hypothesis**

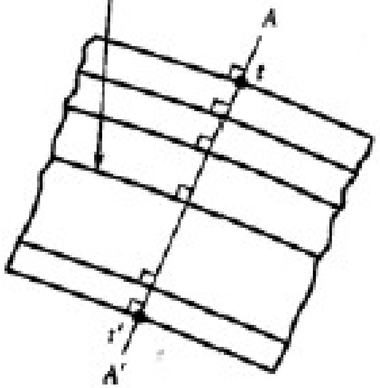
Kirchhoff Hypothesis

Kirchhoff Hypothesis Assumptions
1. Normals remain straight (they do not bend)
2. Normals remain unstretched (they keep the same length)
3. Normals remain normal (they always make a right angle to the neutral plane)

Consequences of Kirchhoff Hypothesis







1. Normals remain straight to the G.M.
2. Normals remain unstretched (no length change in AA')
3. Normals remain normal to the G.M.

Kirchhoff & Higher Order Theories

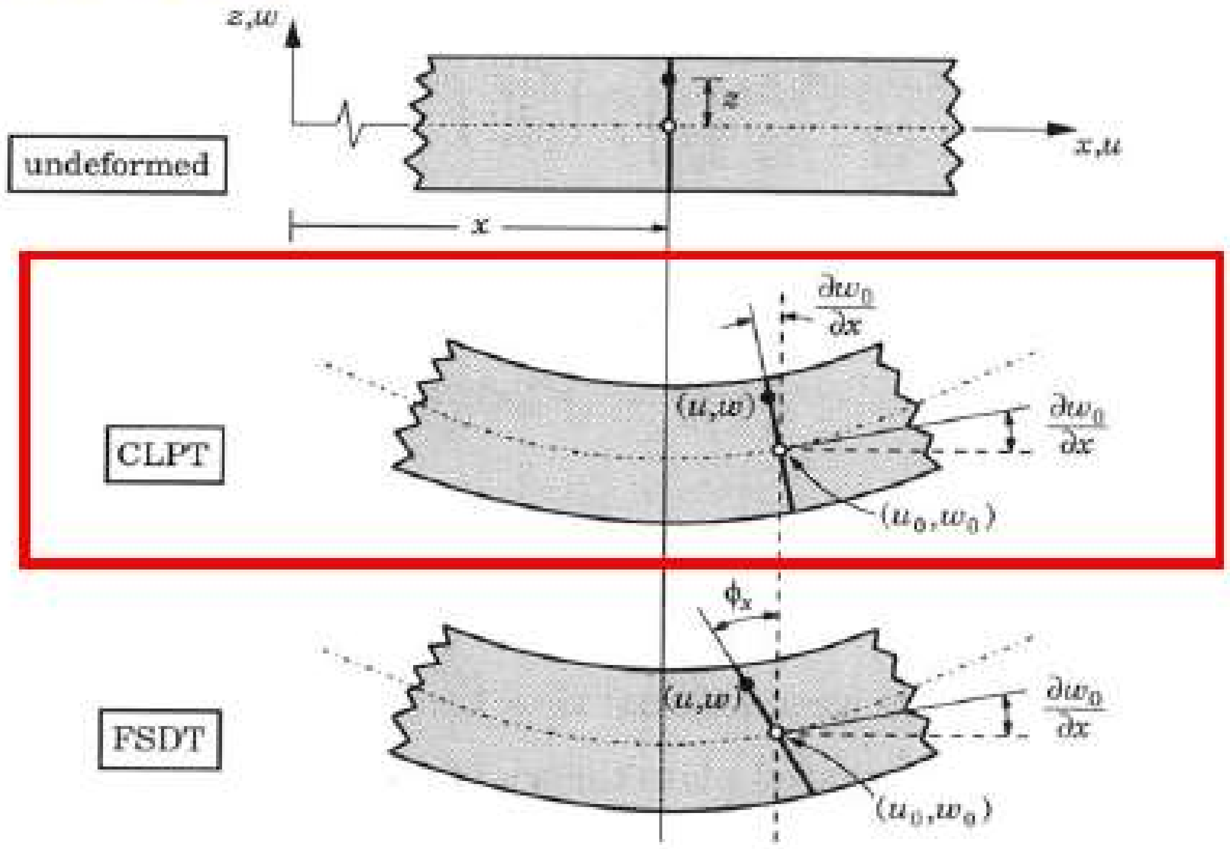
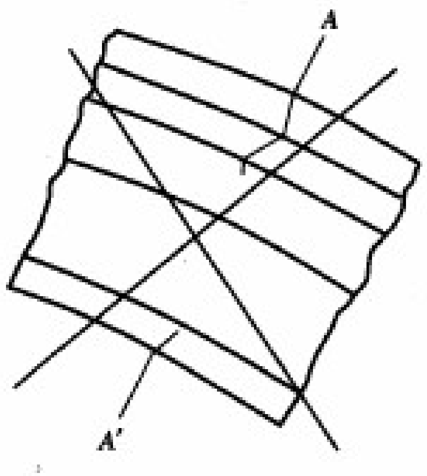


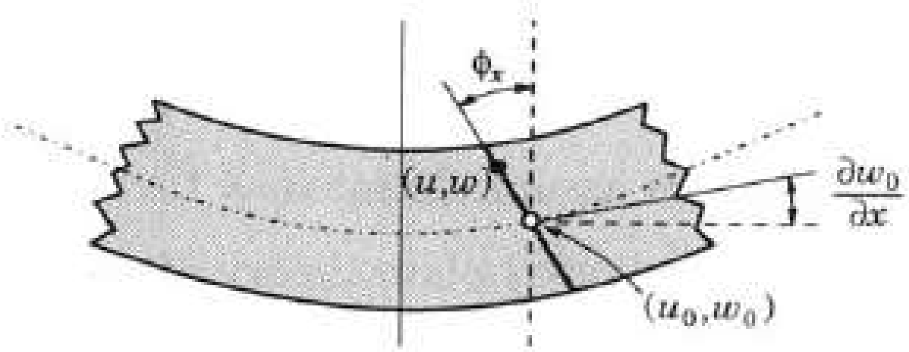
Figure from Reddy, J.N. Mechanics of Laminated Composite Plates, CRC Press, Inc: New York, 1997.

Beyond Kirchhoff Hypothesis

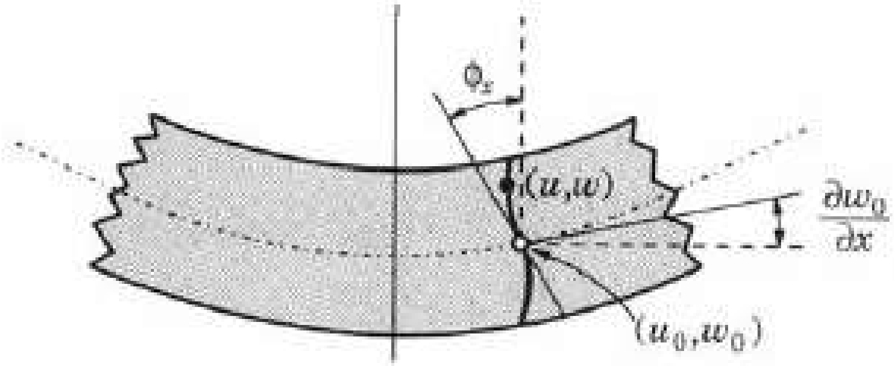


- Kirchhoff **does not** account for shear within the layer
- First order Shear Deformable Theory (FSDT) (Timoshenko) kinematics accounts for shear deformation in the layer

Beyond Kirchhoff Hypothesis

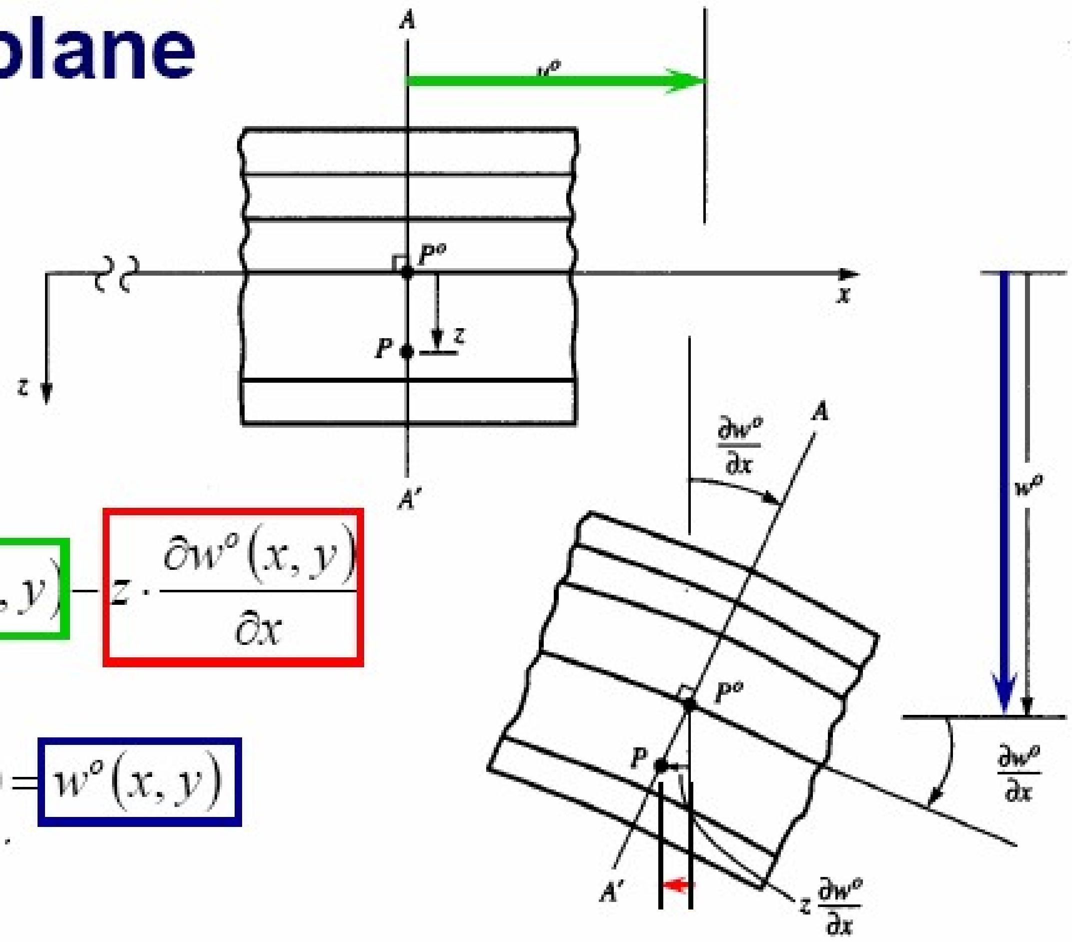


First order Timoshenko kinematics



Higher order beam/plate kinematics allow for the warping in the ply

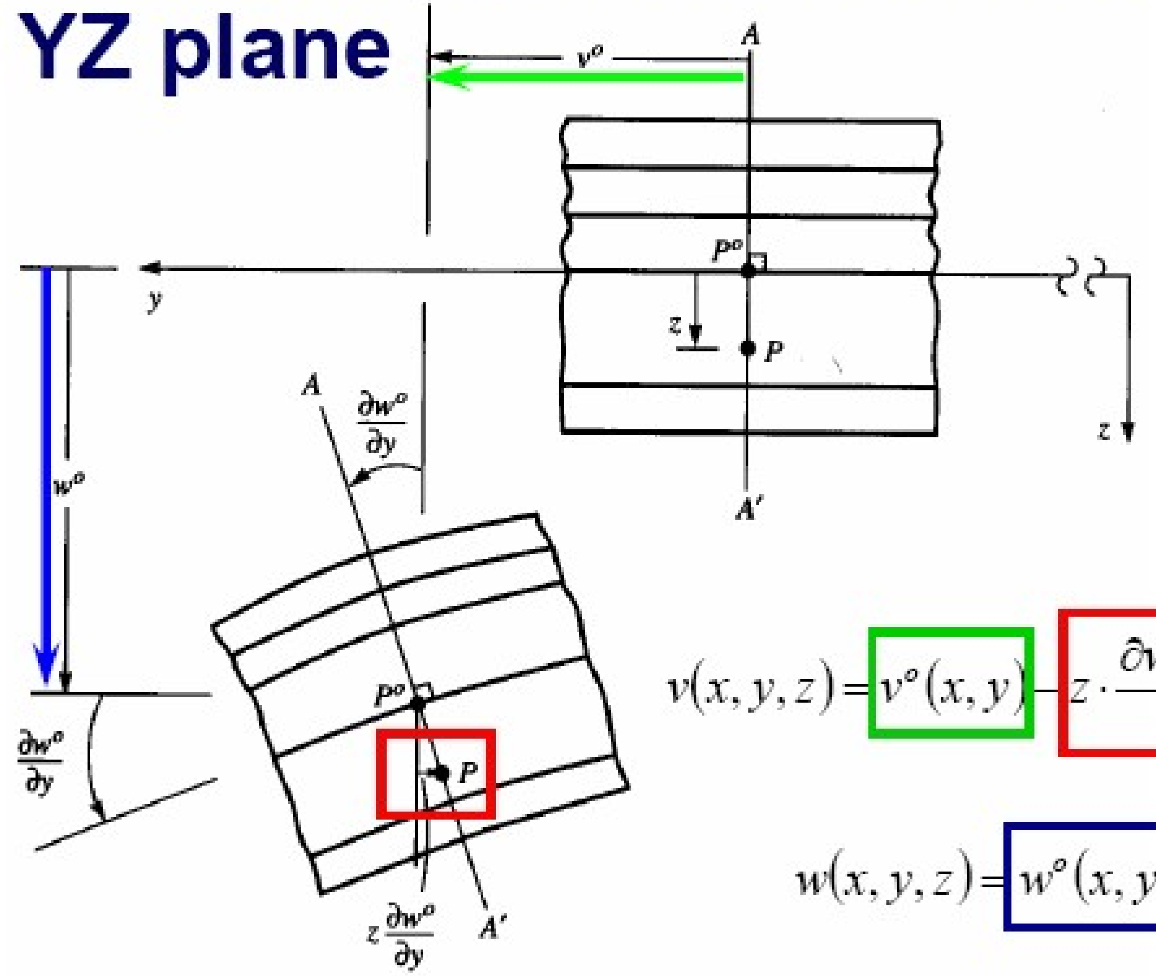
Resulting Displacement Field in XZ plane



$$u(x,y,z)=\boxed{u^o(x,y)}-\boxed{z\cdot\frac{\partial w^o(x,y)}{\partial x}}$$

$$w(x,y,z)=\boxed{w^o(x,y)}$$

Resulting Displacement Field in YZ plane



$$v(x,y,z) = \boxed{v^o(x,y)} - \boxed{z \cdot \frac{\partial w^o(x,y)}{\partial y}}$$

$$w(x,y,z) = \boxed{w^o(x,y)}$$

Displacements to Strains

Kirchhoff Kinematics…

$$u(x,y,z) = u^{o}(x,y) - z \cdot \frac{\partial w^{o}(x,y)}{\partial x} \qquad v(x,y,z) = v^{o}(x,y) - z \cdot \frac{\partial w^{o}(x,y)}{\partial y}$$

$$w(x,y,z) = w^{o}(x,y)$$

Recall the strain displacement relations from the Theory of Elasticity

$$\varepsilon_x = \partial u / \partial x \qquad \gamma_{xy} = \partial v / \partial x + \partial u / \partial y$$

$$\varepsilon_y = \partial v / \partial y \qquad \gamma_{yz} = \partial v / \partial z + \partial w / \partial y$$

$$\varepsilon_z = \partial w / \partial z \qquad \gamma_{zx} = \partial w / \partial x + \partial u / \partial z$$

Laminate Strains

$$\varepsilon_x(x,y,z) \equiv \frac{\partial u(x,y,z)}{\partial x} = \frac{\partial u^o(x,y)}{\partial x} - z \cdot \frac{\partial^2 w^o(x,y)}{\partial x^2}$$

$$\varepsilon_y(x,y,z) \equiv \frac{\partial v(x,y,z)}{\partial y} = \frac{\partial v^o(x,y)}{\partial y} - z \cdot \frac{\partial^2 w^o(x,y)}{\partial y^2}$$

$$\varepsilon_z(x,y,z) \equiv \frac{\partial w(x,y,z)}{\partial z} = \frac{\partial w^o(x,y)}{\partial z} = 0$$

$$\gamma_{yz}(x,y,z) \equiv \frac{\partial w(x,y,z)}{\partial y} + \frac{\partial v(x,y,z)}{\partial z} = \frac{\partial w^o(x,y)}{\partial y} - \frac{\partial w^o(x,y)}{\partial y} = 0$$

$$\gamma_{xz}(x,y,z) \equiv \frac{\partial w(x,y,z)}{\partial x} + \frac{\partial u(x,y,z)}{\partial z} = \frac{\partial w^o(x,y)}{\partial x} - \frac{\partial w^o(x,y)}{\partial x} = 0$$

$$\gamma_{xy}(x,y,z) \equiv \frac{\partial v(x,y,z)}{\partial x} + \frac{\partial u(x,y,z)}{\partial y} = \frac{\partial v^o(x,y)}{\partial x} + \frac{\partial u^o(x,y)}{\partial y} - 2 \cdot z \cdot \frac{\partial^2 w^o(x,y)}{\partial x \cdot \partial y}$$

Note the inconsistency with the constitutive relations developed from the plane stress assumption

Midplane Strains & Curvatures

Small displacement w in the transverse direction ($w << t$)

Small rotation of the reference surface

$$\varepsilon_x^o(x,y) = \frac{\partial u^o(x,y)}{\partial x} \qquad \kappa_x^o(x,y) = -\frac{\partial^2 w^o(x,y)}{\partial x^2}$$

$$\varepsilon_y^o(x,y) = \frac{\partial v^o(x,y)}{\partial y} \qquad \kappa_y^o(x,y) = -\frac{\partial^2 w^o(x,y)}{\partial y^2}$$

$$\gamma_{xy}^o(x,y) = \frac{\partial v^o(x,y)}{\partial x} + \frac{\partial u^o(x,y)}{\partial y} \qquad \kappa_{xy}^o(x,y) = -2 \cdot \frac{\partial^2 w^o(x,y)}{\partial x \cdot \partial y}$$

Defines the strain at any point within the laminate given that we know the reference surface strains and curvatures within the x, y plane.

Laminate Strains using Revised Notation

$$\varepsilon_x(x,y,z) = \varepsilon_x^o(x,y) + z \cdot \kappa_x^o(x,y)$$

$$\varepsilon_y(x,y,z) = \varepsilon_y^o(x,y) + z \cdot \kappa_y^o(x,y)$$

$$\varepsilon_z(x,y,z) = 0$$

$$\gamma_{yz}(x,y,z) = 0$$

$$\gamma_{xz}(x,y,z) = 0$$

$$\gamma_{xy}(x,y,z) = \gamma_{xy}^o(x,y) + z \cdot \kappa_{xy}^o(x,y)$$

The γ_{yz}, and γ_{xz} are zero because the Kirchhoff hypothesis assumes that lines perpendicular to the reference surface before deformation remain perpendicular after the deformation; right angles in the thickness direction do not change when the laminate deforms

Classical Lamination Theory

- The four cornerstones of the lamination theory are the **kinematic**, **constitutive**, **force resultant**, and **equilibrium equations**.
- Let's now consider the **Constitutive Relations**

Laminate Stresses

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\varepsilon_x(x,y,z) = \varepsilon_x^o(x,y) + z \cdot \kappa_x^o(x,y)$$
$$\varepsilon_y(x,y,z) = \varepsilon_y^o(x,y) + z \cdot \kappa_y^o(x,y)$$
$$\gamma_{xy}(x,y,z) = \gamma_{xy}^o(x,y) + z \cdot \kappa_{xy}^o(x,y)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x^o + z \cdot \kappa_x^o \\ \varepsilon_y^o + z \cdot \kappa_y^o \\ \gamma_{xy}^o + z \cdot \kappa_{xy}^o \end{Bmatrix}$$

Using the reduced transformed stiffness matrix and the Kirchhoff strain relations, we can determine the in plane stresses in any ply and at any point within the x, y plane defined by the reference surface.

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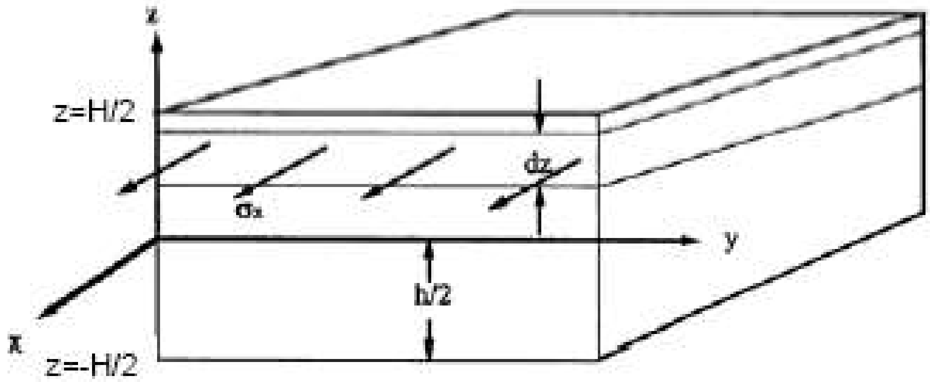
Classical Lamination Theory

- The four cornerstones of the lamination theory are the **kinematic**, **constitutive**, **force resultant**, and **equilibrium equations**.
- Let's now consider the **Force Resultants**

Definitions of Force & Moment Resultants

- Stress in each ply varies through the thickness
- It is convenient to define stresses in terms of equivalent **forces & moments** acting at the middle surface
- Stresses at the edge can be broken into increments and summed
- The resulting integral is defined as the stress resultant, $\mathbf{N}_{ij}$ & $\mathbf{M}_{ij}$ [force or Moment per length]

Stress Resultant in x-dir



$$\text{Total force in x} - \text{direction} = \sum \sigma_x \cdot (dz) \cdot (y)$$

$$\text{As } dz \to 0,\ \sum \sigma_x \cdot (dz) \cdot (\cancel{y}) = \cancel{y} \cdot \int_{-H/2}^{H/2} \sigma_x \cdot dz$$

$$N_x \equiv \int_{-H/2}^{H/2} \sigma_x \cdot dz$$

Stress and Moment Resultants

$$N_x = \int_{-H/2}^{H/2} \sigma_x \cdot dz$$

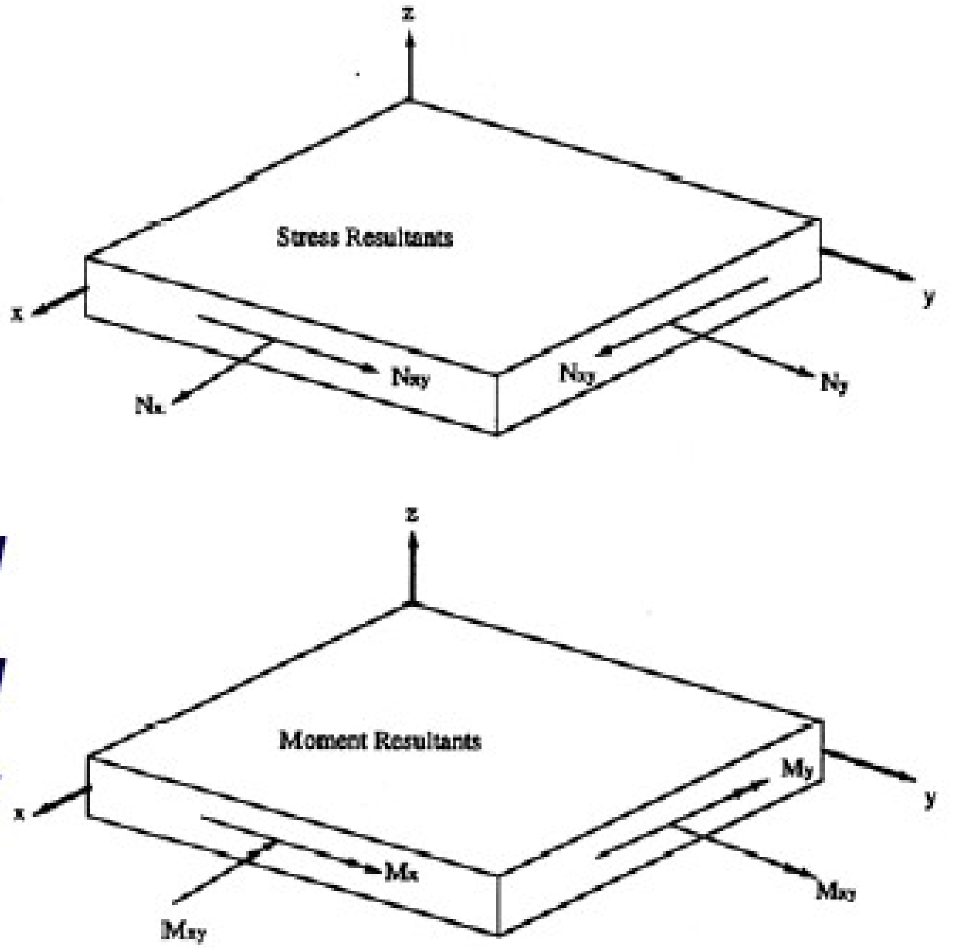
$$N_y = \int_{-H/2}^{H/2} \sigma_y \cdot dz$$

$$N_{xy} = \int_{-H/2}^{H/2} \tau_{xy} \cdot dz$$

$$M_x = \int_{-H/2}^{H/2} \sigma_x \cdot z \cdot dz$$ ***bend***

$$M_y = \int_{-H/2}^{H/2} \sigma_y \cdot z \cdot dz$$ ***bend***

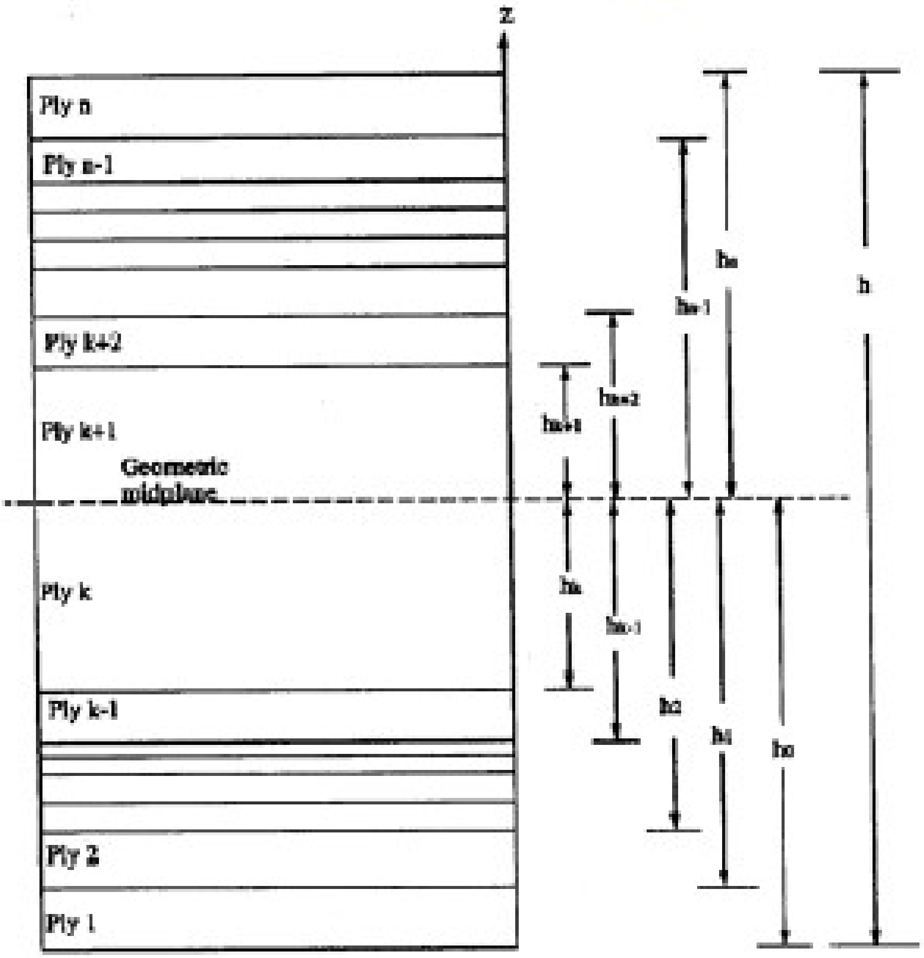
$$M_{xy} = \int_{-H/2}^{H/2} \tau_{xy} \cdot z \cdot dz$$ ***twist***



Putting the Resultants in Matrix Form and Summing

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=0}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k \cdot dz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=0}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k \cdot z \cdot dz$$



Relating Stress to Strain

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=0}^{n} \left\{ \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \cdot \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} \cdot dz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \cdot \begin{bmatrix} \kappa_x^o \\ \kappa_y^o \\ \kappa_{xy}^o \end{bmatrix} \cdot z \cdot dz \right\}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=0}^{n} \left\{ \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \cdot \begin{bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} \cdot z \cdot dz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \cdot \begin{bmatrix} \kappa_x^o \\ \kappa_y^o \\ \kappa_{xy}^o \end{bmatrix} \cdot z^2 \cdot dz \right\}$$

Relationship between mid-plane strains and curvatures, and the resulted force and moment resultants acting on the laminate

Where:

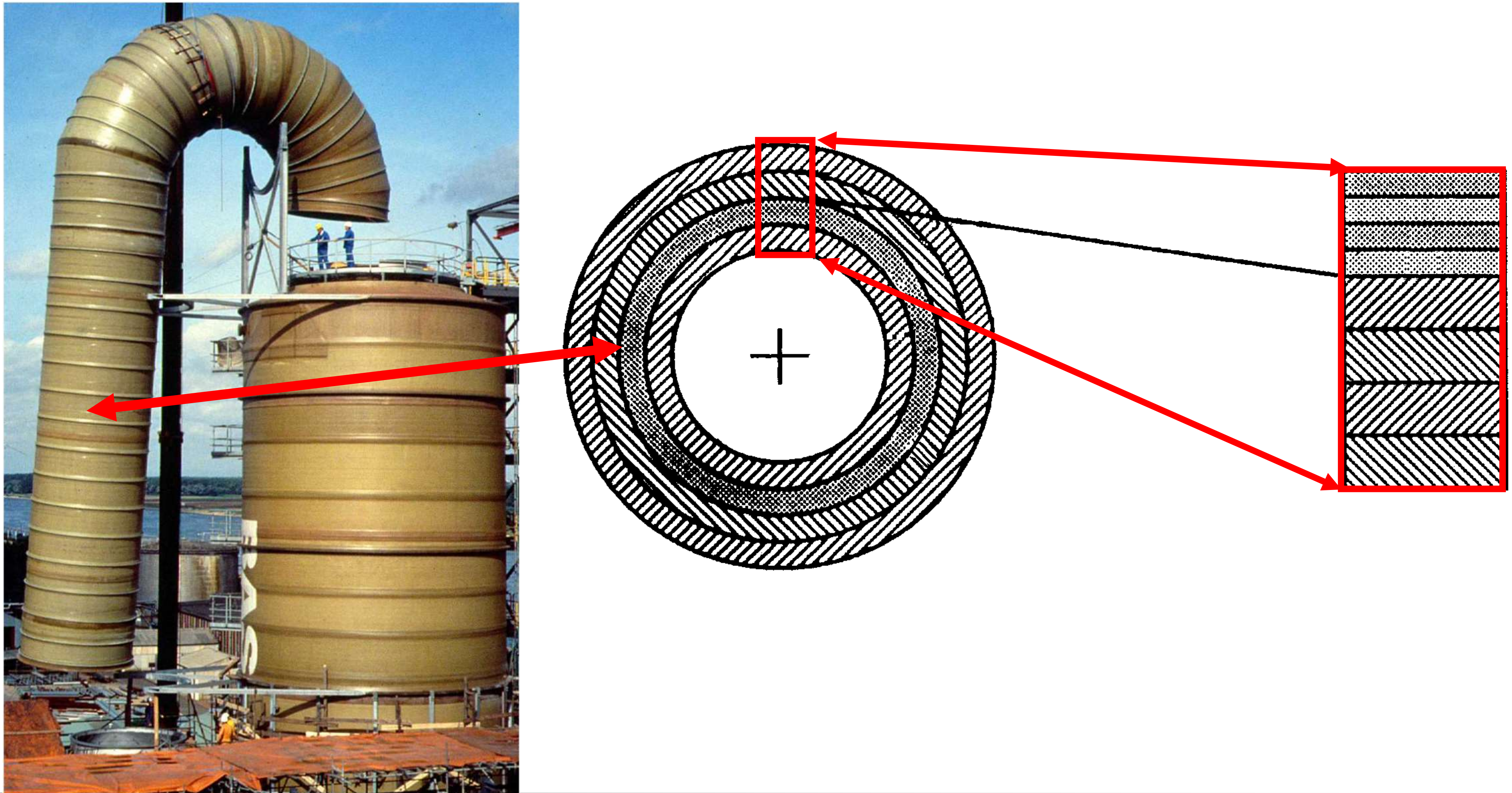
$$\begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{14} & B_{15} & B_{16} \\ A_{21} & A_{22} & A_{23} & B_{24} & B_{25} & B_{26} \\ A_{31} & A_{32} & A_{33} & B_{34} & B_{35} & B_{36} \\ \hline B_{41} & B_{42} & B_{43} & D_{11} & D_{12} & D_{13} \\ B_{51} & B_{52} & B_{53} & D_{21} & D_{22} & D_{23} \\ B_{61} & B_{62} & B_{63} & D_{31} & D_{32} & D_{33} \end{array}\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$[A]$: Extensional Stiffnesses

$[B]$: Coupling Stiffnesses

$[D]$: Bending Stiffnesses

„Small“ Element

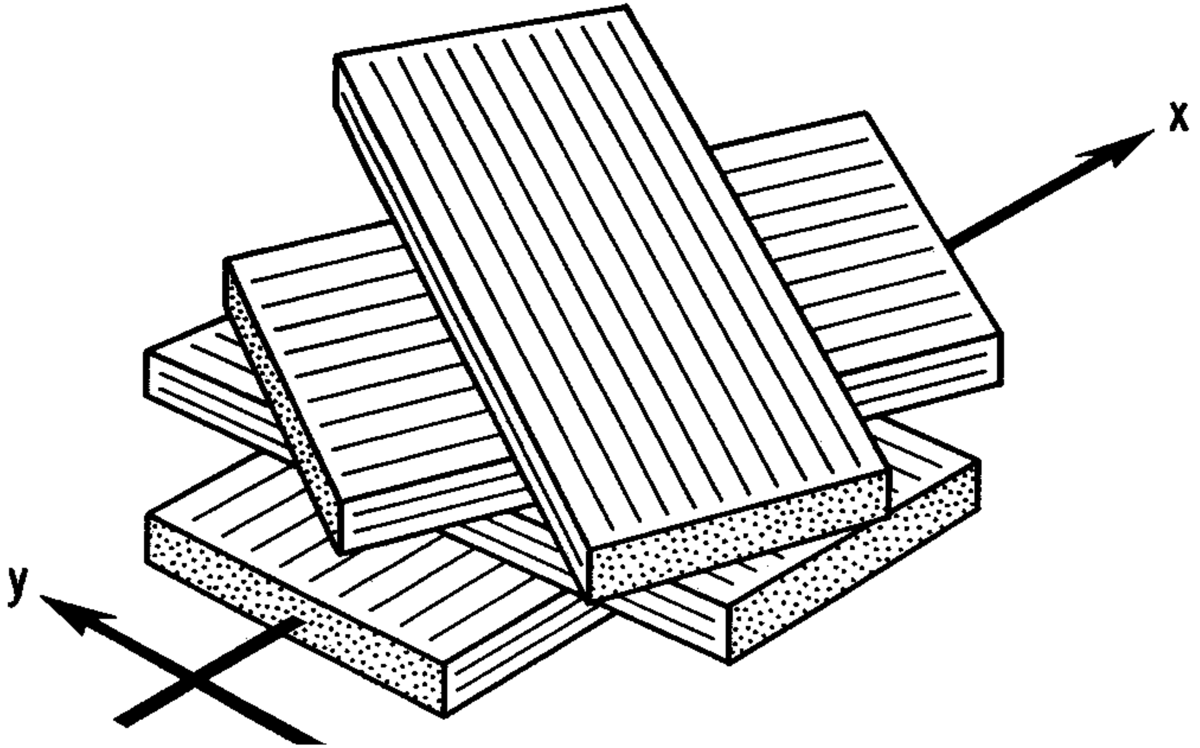


Relationship between force and moment resultants acting on the laminate, and the resulted mid-plane strains and curvatures

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right] \begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \end{bmatrix}$$

$$\varepsilon_x^0 = A^*_{11}\, n_x + A^*_{12}\, n_y + A^*_{13}\, n_{xy} + B^*_{14}\, m_x + B^*_{15}\, m_y + B^*_{16}\, m_{xy}$$

Classical Laminate Theory



Steps for stress analysis in classical laminate theory:

Stress analysis steps can be summarized as follow:

1. Material Parameter as Initial Input

$$E_F,\ \nu_F,\ E_M,\ \nu_M,\ \Phi$$

2. Elasticity Constants of a UD-Lamina

$$E_1^k$$

$$E_2^k$$

$$G_{12}^k$$

$$\nu_{12}^k, \nu_{21}^k$$

3. Stiffnesses of a UD-Lamina

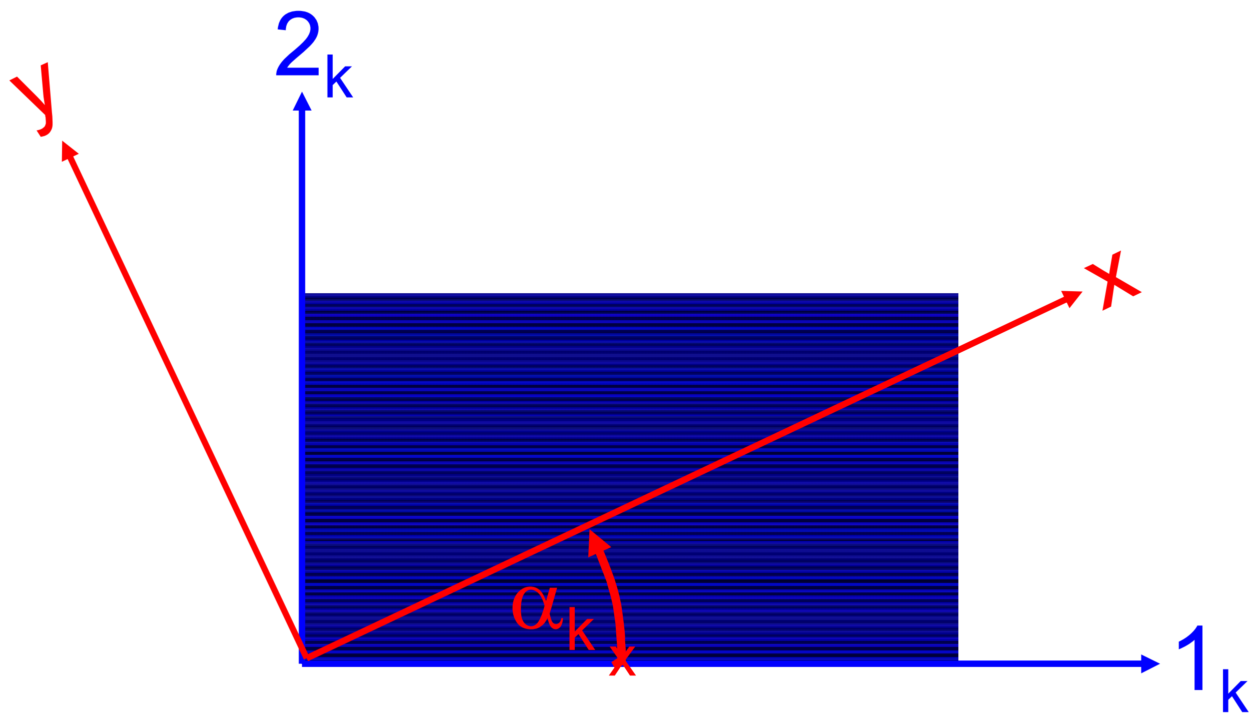
with respect to local 1-2 coordinate system

$$Q_{11}^{k}$$

$$Q_{22}^{k}$$

$$Q_{66}^{k}$$

$$Q_{12}^{k}$$

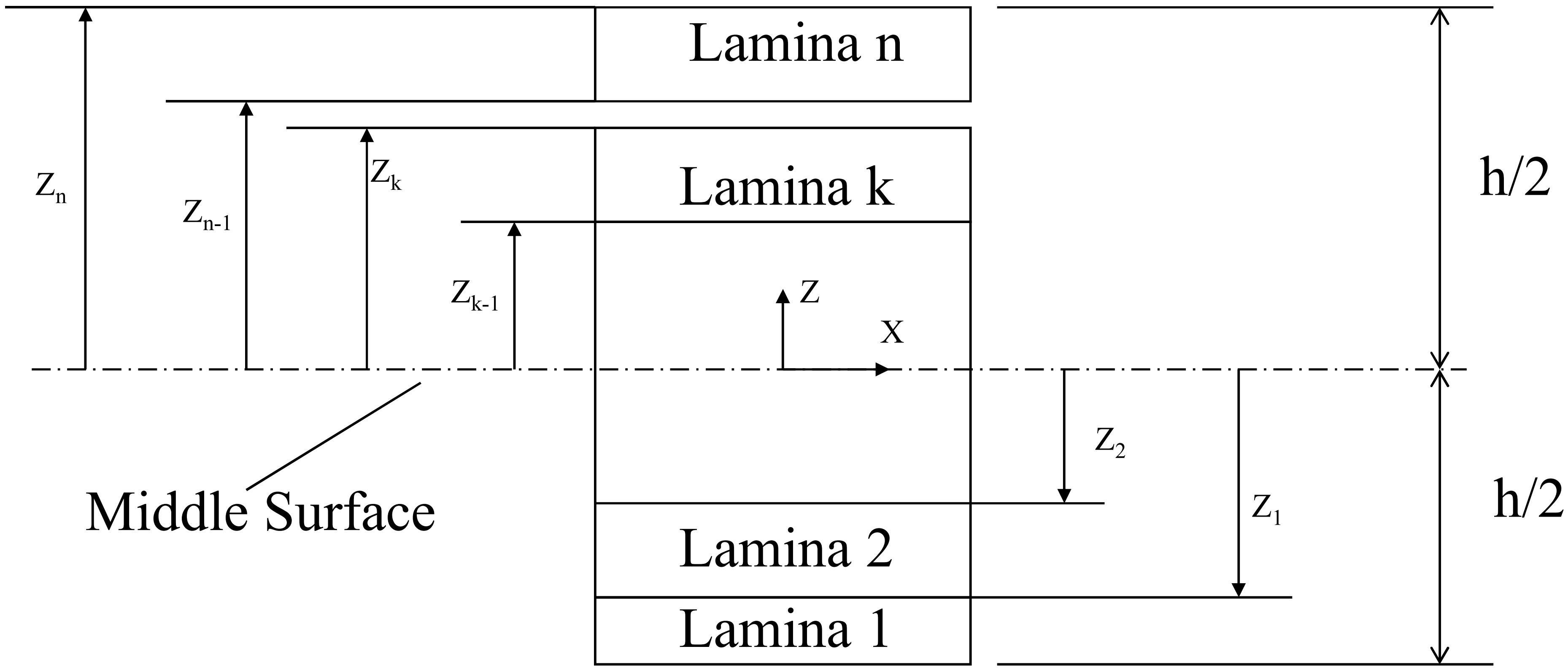


4. Transformed Stiffness Matrix

of a UD-Lamina with respect to the global x-y coordinate system

$$\begin{bmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{13}^k} \\ \overline{Q_{21}^k} & \overline{Q_{22}^k} & \overline{Q_{23}^k} \\ \overline{Q_{31}^k} & \overline{Q_{32}^k} & \overline{Q_{33}^k} \end{bmatrix}$$

Laminate Geometry



5. Extensional Stiffnesses

of a UD-Lamina

$$\left[A\right]_k = \left[\overline{Q}\right]^k h_k$$

$$where \quad h_k :$$ Lamina Thickness = Z_k-Z_{k-1}

6. Bending Stiffnesses

of a UD-Lamina

$$[D]^k$$

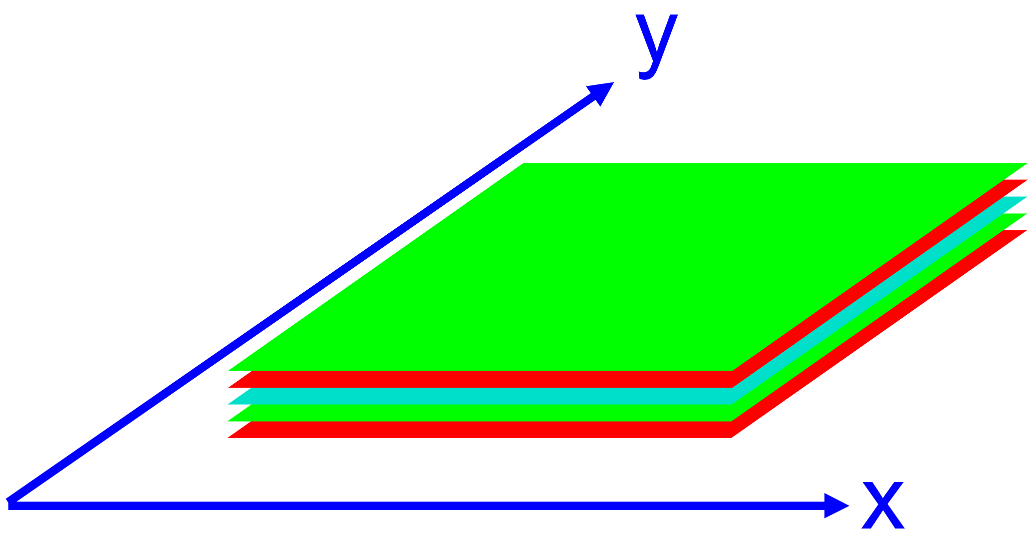
7. Stiffnesses of a Laminate

with respect to x-y coordinate system

$$A_{ij}$$

$$B_{ij}$$

$$D_{ij}$$



Relationship between mid-plane strains and curvatures, and the resulted force and moment resultants acting on the laminate

$$\begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{14} & B_{15} & B_{16} \\ A_{21} & A_{22} & A_{23} & B_{24} & B_{25} & B_{26} \\ A_{31} & A_{32} & A_{33} & B_{34} & B_{35} & B_{36} \\ \hline B_{41} & B_{42} & B_{43} & D_{11} & D_{12} & D_{13} \\ B_{51} & B_{52} & B_{53} & D_{21} & D_{22} & D_{23} \\ B_{61} & B_{62} & B_{63} & D_{31} & D_{32} & D_{33} \end{array}\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

Where:

$[A]$: Extensional Stiffnesses

$[B]$: Coupling Stiffnesses

$[D]$: Bending Stiffnesses

Relationship between force and moment resultants acting on the laminate, and the resulted mid-plane strains and curvatures

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \left[\begin{array}{ccc:ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hdashline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right] \begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \end{bmatrix}$$

$$\varepsilon_x^0 = A^*_{11}\, n_x + A^*_{12}\, n_y + A^*_{13}\, n_{xy} + B^*_{14}\, m_x + B^*_{15}\, m_y + B^*_{16}\, m_{xy}$$

7. Compliances of a Laminate

with respect to x-y coordinate system

$$A^*_{ij}$$

$$B^*_{ij}$$

$$D^*_{ij}$$

8. Elasticity Constants
(for example: calculation of buckling load)

$$\hat{E}_x$$

$$\hat{E}_y$$

$$\hat{G}_{xy}$$

9. Mid-plane strains and curvatures of the laminate

and all UD-Laminas with respect to the x-y coordinate system

$$\varepsilon_x^0 \quad , \quad \varepsilon_y^0 \quad , \quad \gamma_{xy}^0$$

$$\kappa_x \quad , \quad \kappa_y \quad , \quad \kappa_{xy}$$

10. Stresses in the UD-Laminas

with respect to the local UD coordinate system

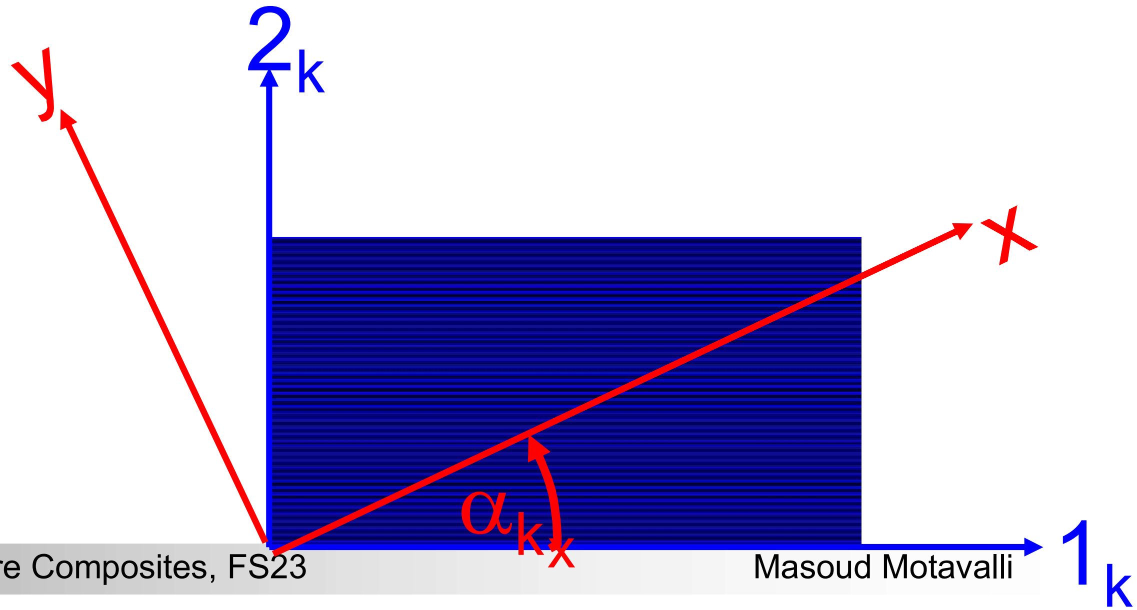
$$\sigma_1^k \qquad \sigma_2^k \qquad \tau_{12}^k$$

Stress verification in every UD-Lamina

Classical Laminate Theory

Stiffnesses of the UD-Laminas with respect to the global x-y coordinate system of the Laminate (transformation)

$$\overline{Q_{11}^{k}} = Q_{11}^{k}\cos^{4}\alpha_{k} + Q_{22}^{k}\sin^{4}\alpha_{k} + 1/2P_{1}^{k}\sin^{2}2\alpha_{k}$$



$$\overline{Q_{22}^{k}} = Q_{22}^{k}\cos^{4}\alpha_{k} + Q_{11}^{k}\sin^{4}\alpha_{k} + 1/2P_{1}^{k}\sin^{2}2\alpha_{k}$$

$$\overline{Q_{33}^{k}} = Q_{66}^{k}\cos^{4}\alpha_{k} + \frac{1}{4}P_{2}^{k}\sin^{2}2\alpha_{k}$$

$$\overline{Q_{12}^{k}} = \overline{Q_{21}^{k}} = Q_{12}^{k} + 1/4P_{2}^{k}\sin^{2}2\alpha_{k}$$

$$\overline{Q_{13}^{k}} = \overline{Q_{31}^{k}} = 1/2\left[P_2^k \sin^2\alpha_k - \left(Q_{11}^{k} - P_1^k\right)\right]\sin 2\alpha_k$$

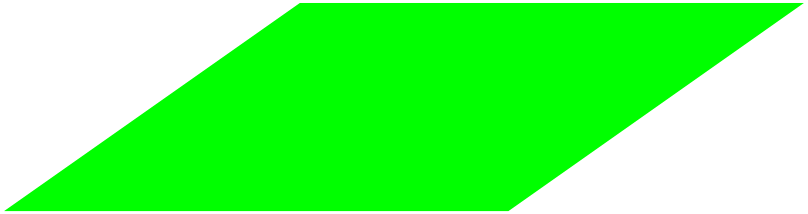
$$\overline{Q_{23}^{k}} = \overline{Q_{32}^{k}} = 1/2\left[-P_2^k \sin^2\alpha_k + \left(Q_{22}^{k} - P_1^k\right)\right]\sin 2\alpha_k$$

$$P_1^k = Q_{12}^k + 2Q_{66}^{k}$$

$$P_2^k = Q_{11}^k + Q_{22}^k - 2P_1^{k}$$

The stiffness matrix of individual laminas in the global x-y coordinate system is:

$$\begin{bmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{13}^k} \\ \overline{Q_{21}^k} & \overline{Q_{22}^k} & \overline{Q_{23}^k} \\ \overline{Q_{31}^k} & \overline{Q_{32}^k} & \overline{Q_{33}^k} \end{bmatrix} \qquad (N/mm^2)$$



where

$$\left[\overline{Q}\right]^k = [T]^k \; [Q]^k \left([T]^k\right)^T$$

and

$$[T]^k = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \qquad with \quad m = \cos\alpha_k \qquad n = \sin\alpha_k$$

The extensional matrix of the Lamina k can be obtained considering its thickness h_k

$$[A]^k = \left[\overline{Q}\right]^k h_k$$

$where \quad h_k :$ Lamina Thickness = $Z_k - Z_{k-1}$

$$[A]^k = \begin{bmatrix} A_{11}^k & A_{12}^k & A_{13}^k \\ A_{21}^k & A_{22}^k & A_{23}^k \\ A_{31}^k & A_{32}^k & A_{33}^k \end{bmatrix}$$

Contribution to the Laminate Stiffness Matrix:

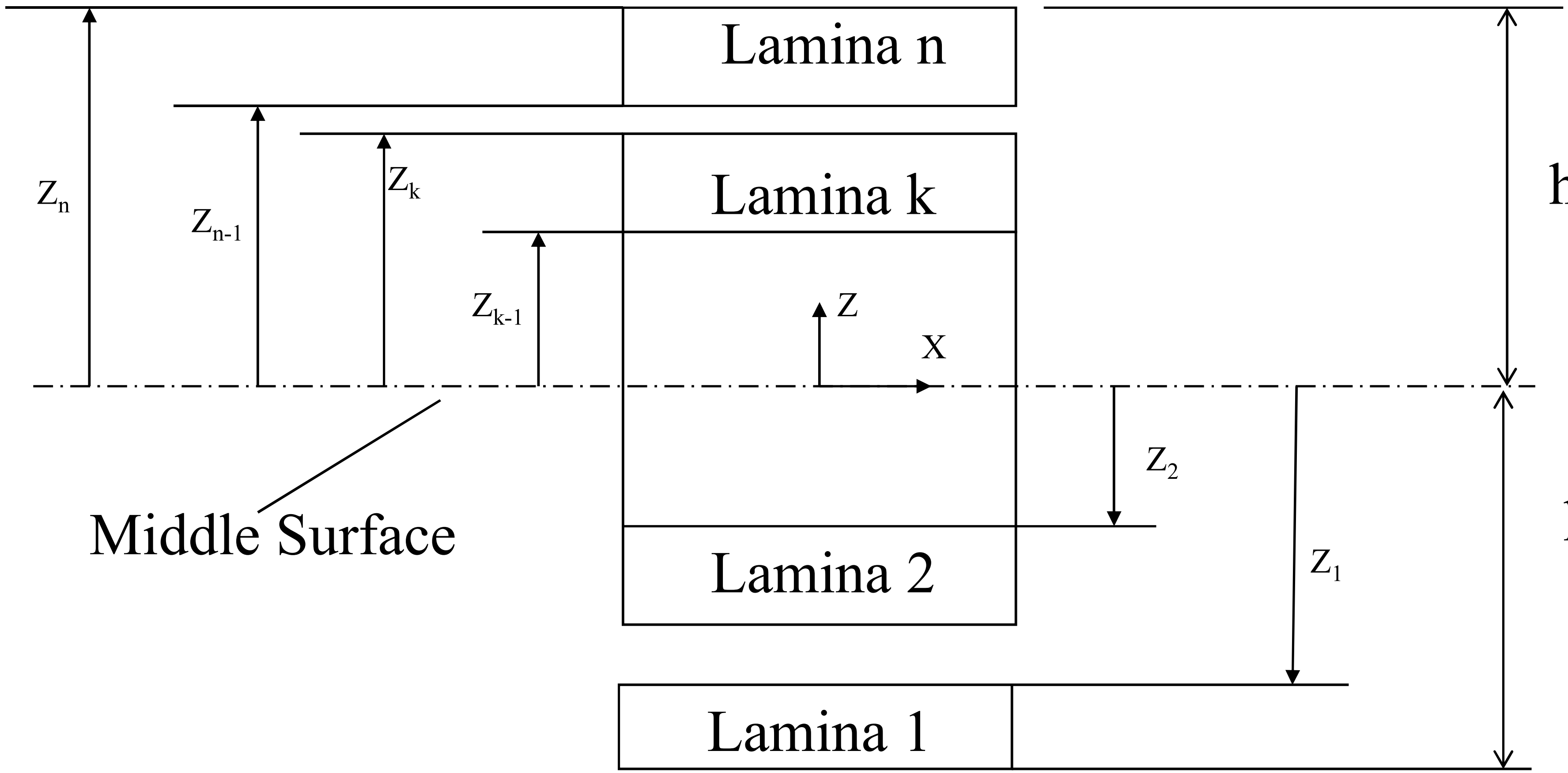
$$\left[\begin{array}{ccc|ccc} \boxed{\begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix}} & & & B_{14} & B_{15} & B_{16} \\ & & & B_{24} & B_{25} & B_{26} \\ & & & B_{34} & B_{35} & B_{36} \\ \hline B_{41} & B_{42} & B_{43} & D_{11} & D_{12} & D_{13} \\ B_{51} & B_{52} & B_{53} & D_{21} & D_{22} & D_{23} \\ B_{61} & B_{62} & B_{63} & D_{31} & D_{32} & D_{33} \end{array}\right]$$

The bending stiffnesses of the UD-Laminas with respect to the global x-y coordinate system can be calculated as follows:

$$[D]^k = \frac{1}{3}\left[\overline{Q}\right]^k \left[z_k^{\,3} - z_{k-1}^{\,3}\right]$$

Laminate Geometry

Be compatible with the coordinate system and the figure: Zk = Z2 and Zk-1 = 1, …



Contribution to the Laminate Stiffness Matrix:

$$[D]^k = \begin{bmatrix} D_{11}^k & D_{12}^k & D_{13}^k \\ D_{21}^k & D_{22}^k & D_{23}^k \\ D_{31}^k & D_{32}^k & D_{33}^k \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} A_{11} & A_{12} & A_{13} & B_{14} & B_{15} & B_{16} \\ A_{21} & A_{22} & A_{23} & B_{24} & B_{25} & B_{26} \\ A_{31} & A_{32} & A_{33} & B_{34} & B_{35} & B_{36} \\ \hdashline B_{41} & B_{42} & B_{43} & D_{11} & D_{12} & D_{13} \\ B_{51} & B_{52} & B_{53} & D_{21} & D_{22} & D_{23} \\ B_{61} & B_{62} & B_{63} & D_{31} & D_{32} & D_{33} \end{array}\right]$$

The coupling stiffnesses of the UD-Laminas with respect to the global x-y coordinate system can be calculated as follows:

$$[B]^k = \frac{1}{2}\left[\overline{Q}\right]^k \left[z_k^{\ 2} - z_{k-1}^{\ 2}\right]$$

Contribution to the Laminate Stiffness Matrix:

$$[B]^k = \begin{bmatrix} B_{11}^k & B_{12}^k & B_{13}^k \\ B_{21}^k & B_{22}^k & B_{23}^k \\ B_{31}^k & B_{32}^k & B_{33}^k \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{14} & B_{15} & B_{16} \\ A_{21} & A_{22} & A_{23} & B_{24} & B_{25} & B_{26} \\ A_{31} & A_{32} & A_{33} & B_{34} & B_{35} & B_{36} \\ \hline B_{41} & B_{42} & B_{43} & D_{11} & D_{12} & D_{13} \\ B_{51} & B_{52} & B_{53} & D_{21} & D_{22} & D_{23} \\ B_{61} & B_{62} & B_{63} & D_{31} & D_{32} & D_{33} \end{array}\right]$$

Laminate stiffnesses with respect to the global x-y coordinate system (superposition)

$$A_{ij} = \sum_{k=1}^{n} A_{ij}^{k}$$

$$D_{ij} = \sum_{k=1}^{n} D_{ij}^{k}$$

$$B_{ij} = \sum_{k=1}^{n} B_{ij}^{k}$$

Laminate stiffnesses with respect to the global x-y coordinate system (superposition)

$$A_{ij} = \sum_{k=1}^{n} A_{ij}^{k}$$

$$D_{ij} = \sum_{k=1}^{n} D_{ij}^{k}$$

$$B_{ij} = \sum_{k=1}^{n} B_{ij}^{k}$$

The most important assumption is that, there is a perfect bond between individual laminas.

Laminate Stiffness Matrix

$$S = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{14} & B_{15} & B_{16} \\ A_{21} & A_{22} & A_{23} & B_{24} & B_{25} & B_{26} \\ A_{31} & A_{32} & A_{33} & B_{34} & B_{35} & B_{36} \\ \hline B_{41} & B_{42} & B_{43} & D_{11} & D_{12} & D_{13} \\ B_{51} & B_{52} & B_{53} & D_{21} & D_{22} & D_{23} \\ B_{61} & B_{62} & B_{63} & D_{31} & D_{32} & D_{33} \end{array}\right]$$

Laminate Compliance Matrix

$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Calculation of the Engineering Constants from the Laminate Compliance Matrix

$$\hat{E}_x = \frac{1}{h}\frac{1}{A^*_{11}}$$

$$\hat{E}_y = \frac{1}{h}\frac{1}{A^*_{22}}$$

Laminate Compliance Matrix

$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Calculation of the Engineering Constants from the Laminate Compliance Matrix

$$\hat{G}_{xy} = \frac{1}{h}\frac{1}{A^*_{33}}$$

Laminate Compliance Matrix

$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Calculation of the Engineering Constants from the Laminate Compliance Matrix

$$\hat{\nu}_{yx} = -\frac{A^*_{12}}{A^*_{11}} = -\frac{A^*_{21}}{A^*_{11}}$$

$$\hat{\nu}_{xy} = -\frac{A^*_{12}}{A^*_{22}} = -\frac{A^*_{21}}{A^*_{22}}$$

Laminate Compliance Matrix

$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Calculation of the Engineering Constants from the Laminate Compliance Matrix

$$E_{xb} = \frac{12}{h^3} \frac{1}{D^*_{11}}$$

$$E_{yb} = \frac{12}{h^3} \frac{1}{D^*_{22}}$$

Laminate Compliance Matrix

$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Calculation of the Engineering Constants from the Laminate Compliance Matrix

$$G_{xyb} = \frac{12}{h^3} \frac{1}{D^*_{33}}$$

Laminate Compliance Matrix

$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Calculation of the Engineering Constants from the Laminate Compliance Matrix

$$\nu_{yxb} = -\frac{D^*_{12}}{D^*_{11}} = -\frac{D^*_{21}}{D^*_{11}}$$

$$\nu_{xyb} = -\frac{D^*_{12}}{D^*_{22}} = -\frac{D^*_{21}}{D^*_{22}}$$

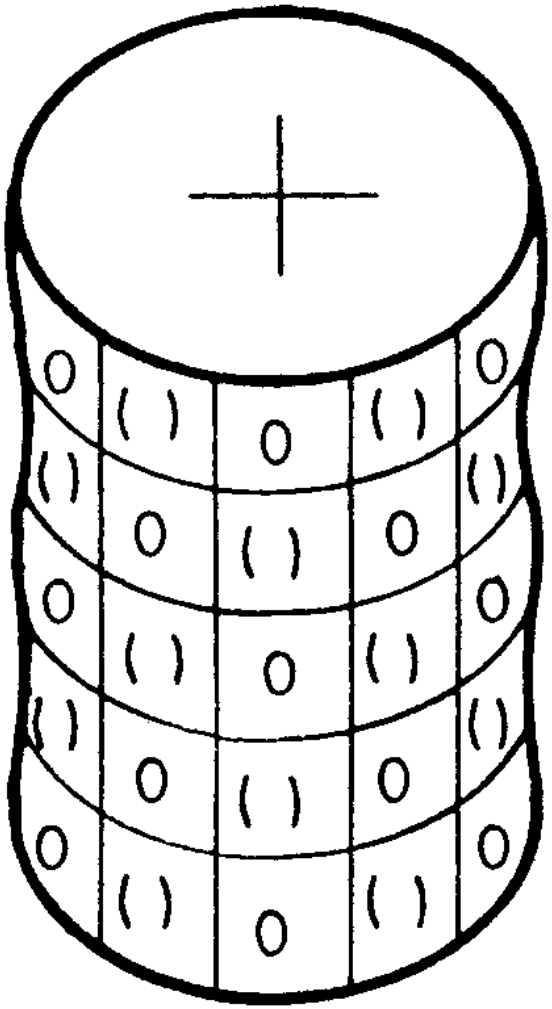
Laminate Compliance Matrix

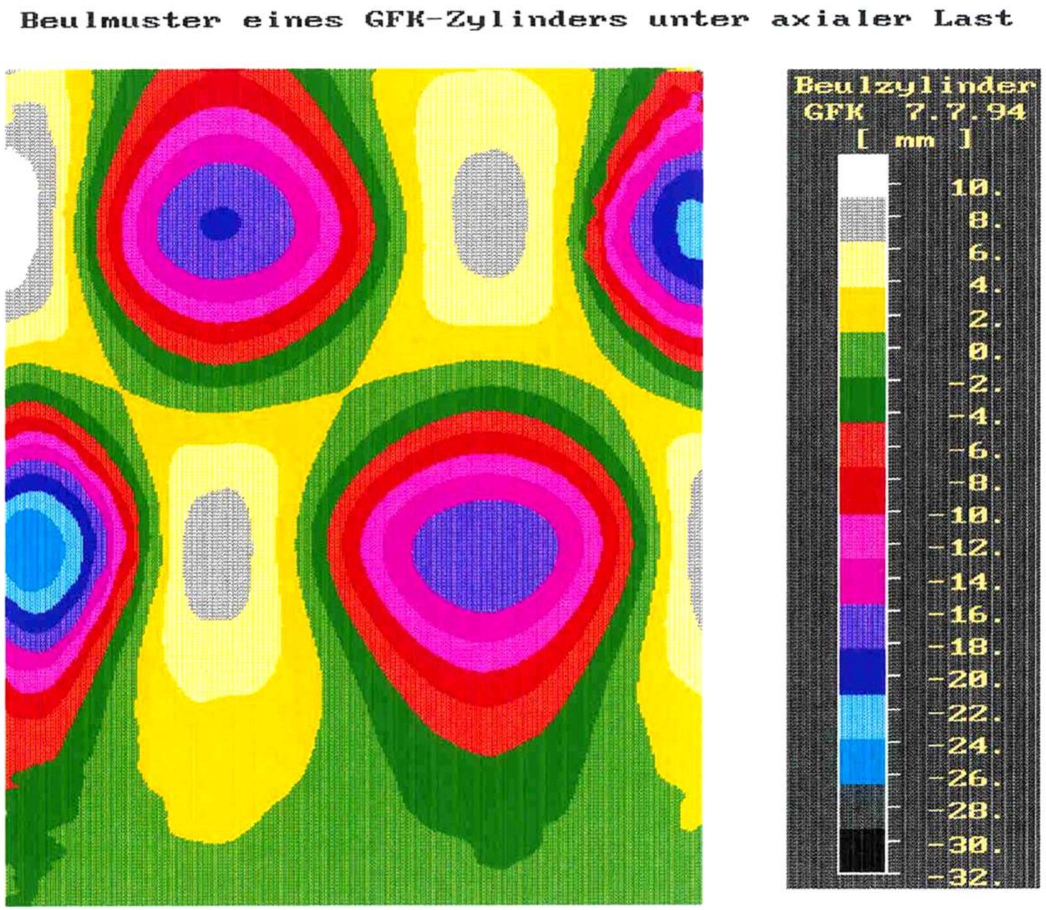
$$S^* = \left[\begin{array}{ccc|ccc} A^*_{11} & A^*_{12} & A^*_{13} & B^*_{14} & B^*_{15} & B^*_{16} \\ A^*_{21} & A^*_{22} & A^*_{23} & B^*_{24} & B^*_{25} & B^*_{26} \\ A^*_{31} & A^*_{32} & A^*_{33} & B^*_{34} & B^*_{35} & B^*_{36} \\ \hline B^*_{41} & B^*_{42} & B^*_{43} & D^*_{11} & D^*_{12} & D^*_{13} \\ B^*_{51} & B^*_{52} & B^*_{53} & D^*_{21} & D^*_{22} & D^*_{23} \\ B^*_{61} & B^*_{62} & B^*_{63} & D^*_{31} & D^*_{32} & D^*_{33} \end{array}\right]$$

Engineering Constants

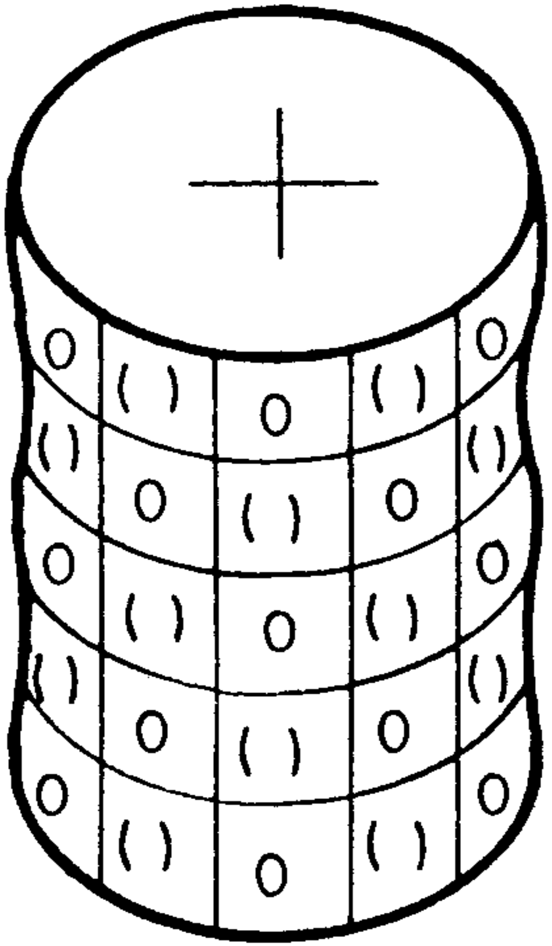
Engineering Constants allow us to apply the equations of classical mechanics (calculation of critical buckling load, etc.)

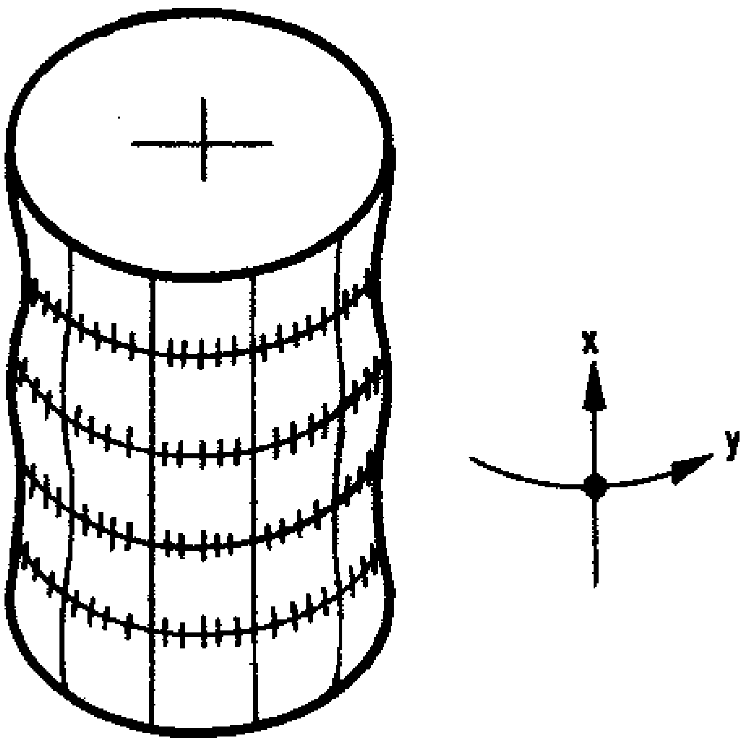
Chessboard buckling of a composite cylindrical shell (Laminate)





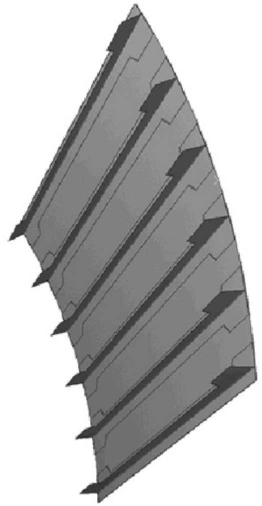
Chessboard buckling





a) Ring Buckling

b) Chessboard Buckling



$$\sigma_{v_R} = \frac{2}{\sqrt{12}} \cdot \frac{t}{r} \sqrt{\frac{\hat{E}_x \cdot \hat{E}_y}{1 - \hat{\nu}_{xy} \cdot \hat{\nu}_{yx}}}$$

$$\sigma_{v_S} = \frac{2}{\sqrt{12}} \cdot \frac{t}{r} \sqrt{\frac{2\,\hat{G}_{xy}\sqrt{\hat{E}_x \cdot \hat{E}_y}}{1 - \sqrt{\hat{\nu}_{xy} \cdot \hat{\nu}_{yx}}}}$$

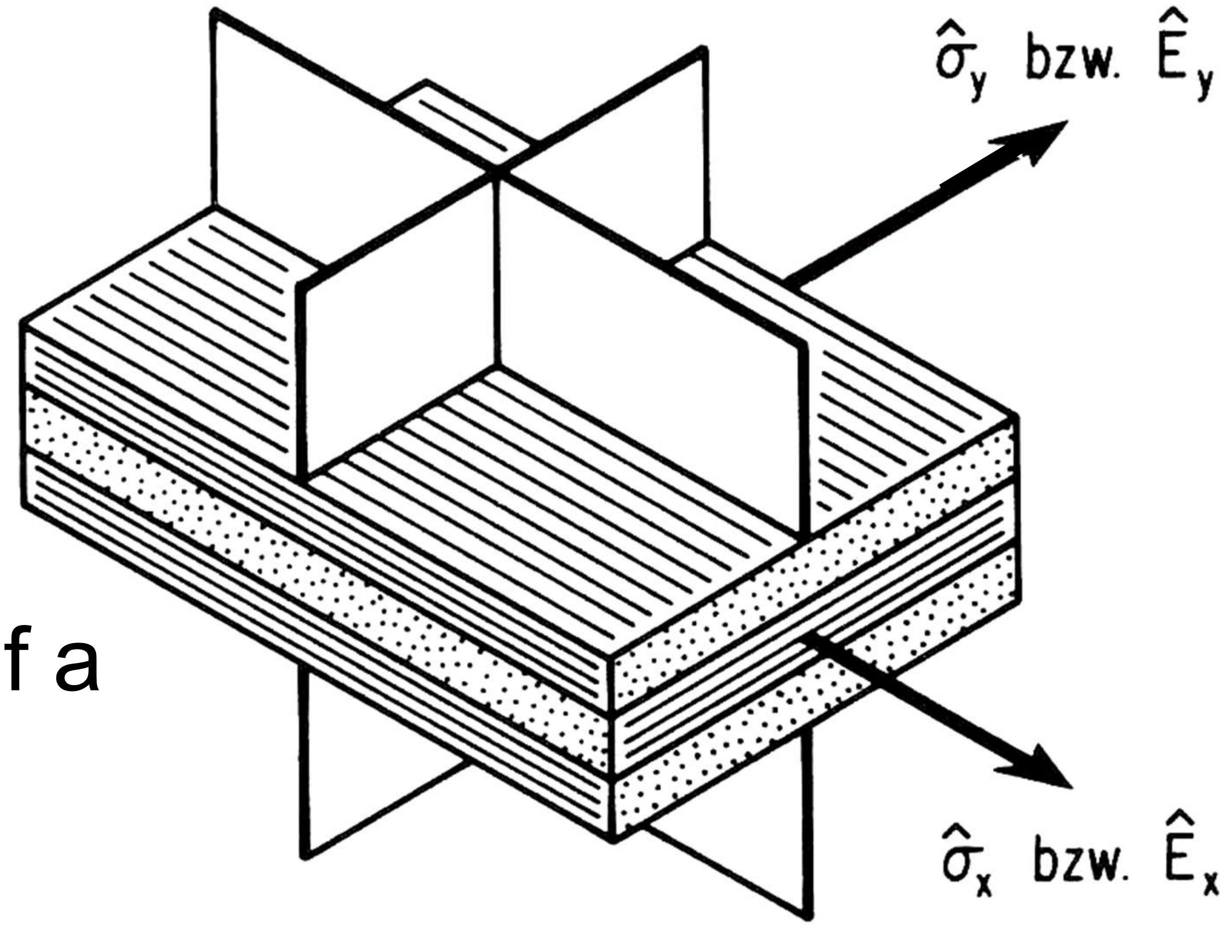
Special case: symmetric unbalanced angle-ply

$$S = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & D_{13} \\ 0 & 0 & 0 & D_{21} & D_{22} & D_{23} \\ 0 & 0 & 0 & D_{31} & D_{32} & D_{33} \end{array}\right]$$

Special case: symmetric cross-ply

$$S = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & \boxed{0} & 0 & 0 & 0 \\ A_{21} & A_{22} & \boxed{0} & 0 & 0 & 0 \\ \boxed{0} & \boxed{0} & A_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & \boxed{0} \\ 0 & 0 & 0 & D_{21} & D_{22} & \boxed{0} \\ 0 & 0 & 0 & \boxed{0} & \boxed{0} & D_{33} \end{array}\right]$$

Special case: cross-ply



Symmetric planes of a laminate

Example:

Non-symmetric crossply laminate with 0° and 90° laminas

Given are:

Crossply a): $(90_8/0_8)$

Crossply b): $(90_4/0_4/90_4/0_4)$

Crossply c): $(90_2/0_2/90_2/0_2/90_2/0_2/90_2/0_2)$

Example continues:

All 16 UD-Laminas have the same thickness ($h_k = h/16$) and are made of type T 300/5208.

Calculate:

Stiffness matrix S of the crossply laminate a) to c)

…Elasticity constants of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
E_1 (N/mm^2)	181'000	204'000	138'000	38'600	76'000
E_2 (N/mm^2)	10'300	18'500	8'960	8'270	5'500
ν_{12}	0.28	0.23	0.30	0.26	0.34
G_{12} (N/mm^2)	7'170	5'590	7'100	4'140	2'300
S_{11} (mm^2/N)	$5.525 \cdot 10^{-6}$	$4.902 \cdot 10^{-6}$	$7.246 \cdot 10^{-6}$	$25.91 \cdot 10^{-6}$	$13.16 \cdot 10^{-6}$
S_{22} (mm^2/N)	$97.09 \cdot 10^{-6}$	$54.05 \cdot 10^{-6}$	$111.6 \cdot 10^{-6}$	$120.9 \cdot 10^{-6}$	$181.8 \cdot 10^{-6}$
S_{12} (mm^2/N)	$-1.547 \cdot 10^{-6}$	$-1.128 \cdot 10^{-6}$	$-2.174 \cdot 10^{-6}$	$-6.744 \cdot 10^{-6}$	$-4.474 \cdot 10^{-6}$
S_{33} (mm^2/N)	$139.5 \cdot 10^{-6}$	$172.7 \cdot 10^{-6}$	$140.8 \cdot 10^{-6}$	$241.5 \cdot 10^{-6}$	$434.8 \cdot 10^{-6}$
Q_{11} (N/mm^2)	181'800	205'000	138'000	39'160	76'640
Q_{22} (N/mm^2)	10'340	18'580	9'013	8'392	5'546
Q_{12} (N/mm^2)	2'897	4'275	2'704	2'182	1'886
Q_{33} (N/mm^2)	7'170	5'790	7'100	4'140	2'300

Crossply laminate a)

$$\underline{\underline{S}} = \left[\begin{array}{ccc|ccc} 192\cdot10^3 & 5.7\cdot10^3 & 0 & \boxed{-85.5\cdot10^3} & 0 & 0 \\ 5.7\cdot10^3 & 192\cdot10^3 & 0 & 0 & \boxed{-85.5\cdot10^3} & 0 \\ 0 & 0 & 14.3\cdot10^3 & 0 & 0 & 0 \\ \hline \boxed{-85.5\cdot10^3} & 0 & 0 & 64\cdot10^3 & 1.9\cdot10^3 & 0 \\ 0 & \boxed{-85.5\cdot10^3} & 0 & 1.9\cdot10^3 & 64\cdot10^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.7\cdot10^3 \end{array}\right]$$

Crossply laminate b)

$$\underline{\underline{S}} = \left[\begin{array}{ccc|ccc} 192\cdot10^3 & 5.7\cdot10^3 & 0 & \boxed{-42.8\cdot10^3} & 0 & 0 \\ 5.7\cdot10^3 & 192\cdot10^3 & 0 & 0 & \boxed{-42.8\cdot10^3} & 0 \\ 0 & 0 & 14.3\cdot10^3 & 0 & 0 & 0 \\ \hline \boxed{-42.8\cdot10^3} & 0 & 0 & 64\cdot10^3 & 1.9\cdot10^3 & 0 \\ 0 & \boxed{-42.8\cdot10^3} & 0 & 1.9\cdot10^3 & 64\cdot10^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.7\cdot10^3 \end{array}\right]$$

Crossply laminate c)

$$\underline{\underline{S}} = \left[\begin{array}{ccc|ccc} 192 \cdot 10^3 & 5.7 \cdot 10^3 & 0 & \boxed{-21.4 \cdot 10^3} & 0 & 0 \\ 5.7 \cdot 10^3 & 192 \cdot 10^3 & 0 & 0 & \boxed{-21.4 \cdot 10^3} & 0 \\ 0 & 0 & 14.3 \cdot 10^3 & 0 & 0 & 0 \\ \hline \boxed{-21.4 \cdot 10^3} & 0 & 0 & 64 \cdot 10^3 & 1.9 \cdot 10^3 & 0 \\ 0 & \boxed{-21.4 \cdot 10^3} & 0 & 1.9 \cdot 10^3 & 64 \cdot 10^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.7 \cdot 10^3 \end{array}\right]$$

Calculation of stresses in the UD-Laminas

$$\begin{bmatrix} \sigma_x^k \\ \sigma_y^k \\ \tau_{xy}^k \end{bmatrix} = \begin{bmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{13}^k} \\ \overline{Q_{21}^k} & \overline{Q_{22}^k} & \overline{Q_{23}^k} \\ \overline{Q_{31}^k} & \overline{Q_{32}^k} & \overline{Q_{33}^k} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 + z_k \kappa_x \\ \varepsilon_y^0 + z_k \kappa_y \\ \gamma_{xy}^0 + z_k \kappa_{xy} \end{bmatrix}$$

Transformation of the lamina stresses from global x-y coordinate into the local 1-2 coordinate system

$$\begin{bmatrix} \sigma_1^k \\ \sigma_2^k \\ \tau_{12}^k \end{bmatrix} = \left([T]^k\right)^{-1} \begin{bmatrix} \sigma_x^k \\ \sigma_y^k \\ \tau_{xy}^k \end{bmatrix}$$

where:

$$[T]^k = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}$$

$$with \quad m = \cos\alpha_k \qquad n = \sin\alpha_k$$

Calculation of lamina strains in the local 1-2 coordinate system:

$$\varepsilon_1^k = \frac{1}{E_1}\sigma_1^k - \frac{\nu_{12}}{E_1}\sigma_2^k$$

$$\varepsilon_2^k = -\frac{\nu_{12}}{E_1}\sigma_1^k + \frac{1}{E_2}\sigma_2^k$$

$$\gamma_{12}^k = \frac{1}{G_{12}}\tau_{12}^k$$

Stress Verification

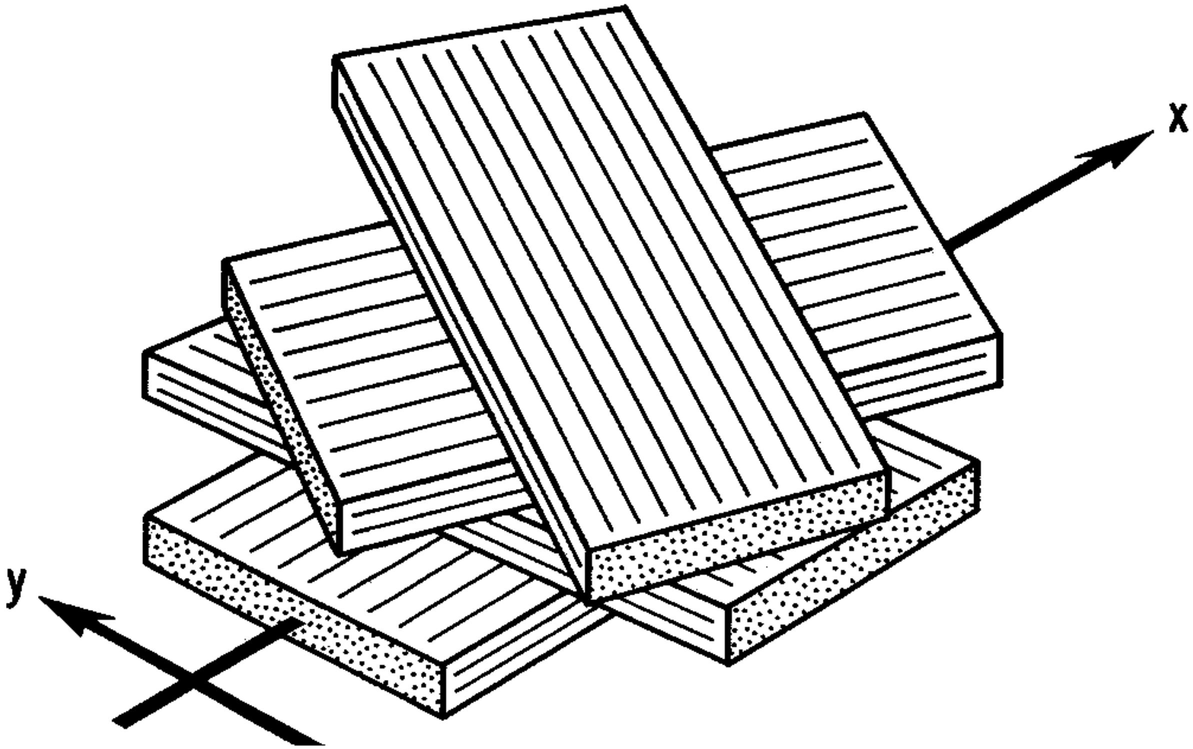
$$\sigma_1^k < \sigma_{1,admissible}^k$$

$$\sigma_2^k < \sigma_{2,admissible}^k$$

$$\tau_{12}^k < \tau_{12,admissible}^k$$

Classical Laminate Theory: Thermal Loading

Classical Laminate Theory



Steps for stress analysis:

Stress analysis steps can be summarized as follow:

Material Parameter as Initial Input

$$E_F,\ \nu_F,\ E_M,\ \nu_M,\ \Phi,\ \alpha_{\mathbf{F}},\ \alpha_{\mathbf{M}}$$

Thermal Loading

$$\Delta T$$

Thermal properties of a UD-Lamina:
Expansion cofficients with respect to the local 1-2 UD-Lamina coordinate system (Book Geoff Eckold p59)

$$\alpha_1 = \alpha_F + \frac{\alpha_M - \alpha_F}{\dfrac{\Phi E_F}{(1-\Phi)E_M} + 1}$$

And (Book Geoff Eckold p59)

$$\alpha_2 = \alpha_F \Phi + \alpha_M (1-\Phi) + \nu_F \alpha_F \Phi + \nu_M \alpha_M (1-\Phi) - [\nu_F \Phi + \nu_M (1-\Phi)]\alpha_1$$

Expansion cofficients with respect to the global x-y coordinate system

$$\begin{pmatrix} \alpha_x^k \\ \alpha_y^k \\ \alpha_{xy}^k \end{pmatrix} = [T]^k \begin{pmatrix} \alpha_1^k \\ \alpha_2^k \\ 0 \end{pmatrix}$$

where

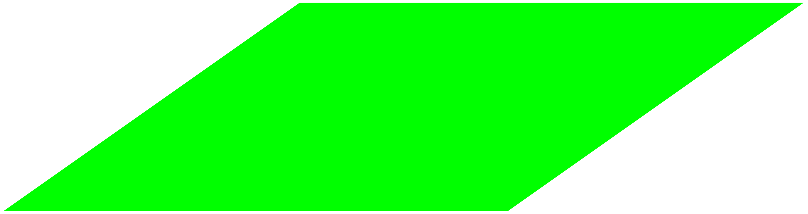
$$[T]^k = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \qquad with \quad m = \cos\alpha_k \qquad n = \sin\alpha_k$$

where:

α_F	=	Thermal expansion coefficient of fibers
α_M	=	Thermal expansion coefficient of matrix
ν_M	=	Poisson's ratio of matrix
ν_F	=	Poisson's ratio of fibers
E_F	=	Elasticity modulus of fibers
E_M	=	Elasticity modulus of matrix
Φ	=	Fiber volume fraction

The stiffness matrix of individual laminas in the global x-y coordinate system is:

$$\begin{bmatrix} \overline{Q_{11}^{k}} & \overline{Q_{12}^{k}} & \overline{Q_{13}^{k}} \\ \overline{Q_{21}^{k}} & \overline{Q_{22}^{k}} & \overline{Q_{23}^{k}} \\ \overline{Q_{31}^{k}} & \overline{Q_{32}^{k}} & \overline{Q_{33}^{k}} \end{bmatrix} \qquad (N / mm^2)$$



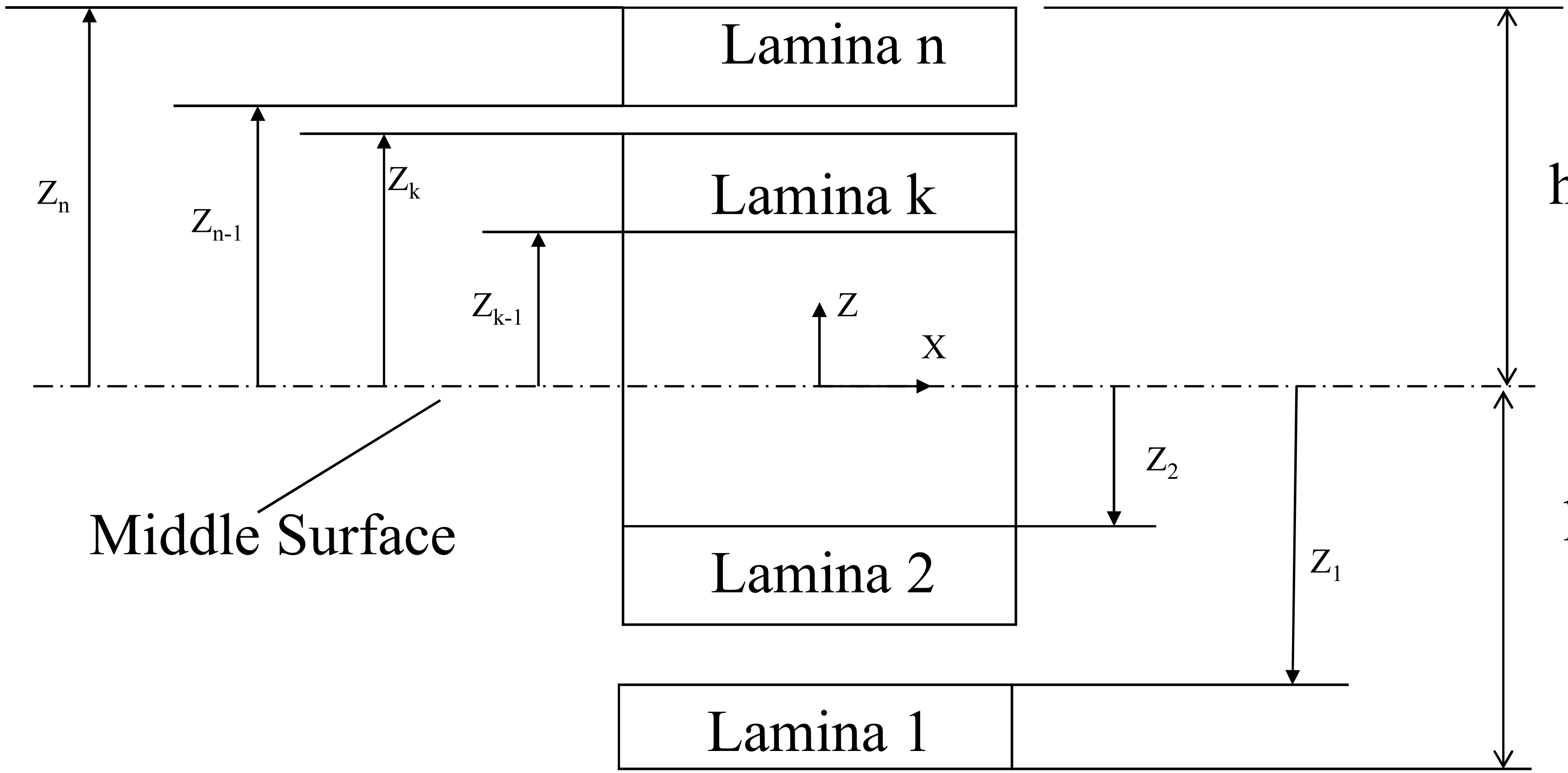
$where$

$$\left[\overline{Q}\right]^k = [T]^k \; [Q]^k ([T]^k)^T$$

and

$$[T]^k = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \qquad with \quad m = \cos\alpha_k \qquad n = \sin\alpha_k$$

Laminate Geometry



Force and moment resultant caused by thermal loading in the k-th UD-Lamina:

$$\begin{pmatrix} {n_x^k}^T \\ {n_y^k}^T \\ {n_{xy}^k}^T \end{pmatrix} = \begin{bmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{13}^k} \\ \overline{Q_{21}^k} & \overline{Q_{22}^k} & \overline{Q_{23}^k} \\ \overline{Q_{31}^k} & \overline{Q_{32}^k} & \overline{Q_{33}^k} \end{bmatrix} \begin{pmatrix} \alpha_x^k . \Delta T . h^k \\ \alpha_y^k . \Delta T . h^k \\ \alpha_{xy}^k . \Delta T . h^k \end{pmatrix}$$

and

$$\begin{pmatrix} {m_x^k}^T \\ {m_y^k}^T \\ {m_{xy}^k}^T \end{pmatrix} = \begin{bmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{13}^k} \\ \overline{Q_{21}^k} & \overline{Q_{22}^k} & \overline{Q_{23}^k} \\ \overline{Q_{31}^k} & \overline{Q_{32}^k} & \overline{Q_{33}^k} \end{bmatrix} \begin{pmatrix} \alpha_x^k . \Delta T . z^k . h^k \\ \alpha_y^k . \Delta T . z^k . h^k \\ \alpha_{xy}^k . \Delta T . z^k . h^k \end{pmatrix}$$

Force and moment resultant caused by thermal loading in the laminate (superposition):

$$\begin{pmatrix} n_x^{\ T} \\ n_y^{\ T} \\ n_{xy}^{\ T} \end{pmatrix} = \sum_{k=1}^{n} \begin{pmatrix} n_x^{k\,T} \\ n_y^{k\,T} \\ n_{xy}^{k\ T} \end{pmatrix}$$

$$and$$

$$\begin{pmatrix} m_x^{\ T} \\ m_y^{\ T} \\ m_{xy}^{\ T} \end{pmatrix} = \sum_{k=1}^{n} \begin{pmatrix} m_x^{k\,T} \\ m_y^{k\,T} \\ m_{xy}^{k\ T} \end{pmatrix}$$

Mid plane strains and curvatures caused by thermal and mechanical loading with respect to the laminate global x-y coordinate system:

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = [S*] \begin{bmatrix} n_x + n_x^T \\ n_y + n_y^T \\ n_{xy} + n_{xy}^T \\ m_x + m_x^T \\ m_y + m_y^T \\ m_{xy} + m_{xy}^T \end{bmatrix}$$

Calculation of stresses in the UD-Laminas

$$\begin{bmatrix} \sigma_x^k \\ \sigma_y^k \\ \tau_{xy}^k \end{bmatrix} = \begin{bmatrix} \overline{Q_{11}^k} & \overline{Q_{12}^k} & \overline{Q_{13}^k} \\ \overline{Q_{21}^k} & \overline{Q_{22}^k} & \overline{Q_{23}^k} \\ \overline{Q_{31}^k} & \overline{Q_{32}^k} & \overline{Q_{33}^k} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 + z_k \kappa_x - \alpha_x^k \Delta T \\ \varepsilon_y^0 + z_k \kappa_y - \alpha_y^k \Delta T \\ \gamma_{xy}^0 + z_k \kappa_{xy} - \alpha_{xy}^k \Delta T \end{bmatrix}$$

Transformation of the lamina stresses from global x-y coordinate into the local 1-2 coordinate system

$$\begin{bmatrix} \sigma_1^k \\ \sigma_2^k \\ \tau_{12}^k \end{bmatrix} = \left([T]^k \right)^{-1} \begin{bmatrix} \sigma_x^k \\ \sigma_y^k \\ \tau_{xy}^k \end{bmatrix}$$

where:

$$[T]^k = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}$$

$$with \qquad m = \cos\alpha_k \qquad n = \sin\alpha_k$$

Calculation of lamina strains in the local 1-2 coordinate system:

$$\varepsilon_1^k = \frac{1}{E_1}\sigma_1^k - \frac{\nu_{12}}{E_1}\sigma_2^k$$

$$\varepsilon_2^k = -\frac{\nu_{12}}{E_1}\sigma_1^k - \frac{1}{E_2}\sigma_2^k$$

$$\gamma_{12}^k = \frac{1}{G_{12}}\tau_{12}^k$$

Stress Verification

$$\sigma_1^k < \sigma_{1,admissible}^k$$

$$\sigma_2^k < \sigma_{2,admissible}^k$$

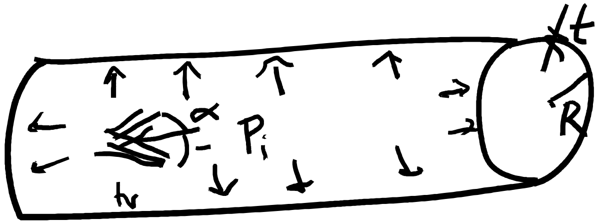
$$\tau_{12}^k < \tau_{12,admissible}^k$$

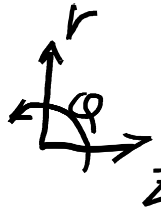
Laminate Theory

$$Q \longrightarrow \bar{Q} \longrightarrow \text{Superposition} \longrightarrow \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

$$\longrightarrow \varepsilon_{global} \longrightarrow \varepsilon_{local} \longrightarrow \sigma_{local}$$

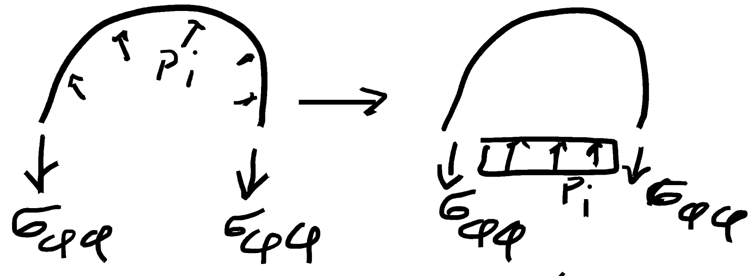
Example ① Pressure Vessel made of FRP





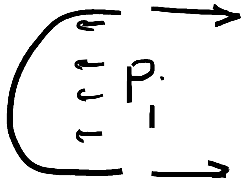
Zlyndrical Coord. syst.

$\frac{t}{R} < 0,1$



$\rightarrow P_i \cdot 2R = \sigma_{\varphi\varphi} \cdot 2t$

$\rightarrow \boxed{\sigma_{\varphi\varphi} = \frac{P_i R}{t}}$



$P_i \cdot \pi R^2 = \sigma_{zz} \cdot 2\pi R t$

$\rightarrow \boxed{\sigma_{zz} = \frac{P_i \cdot R}{2t}}$

$\sigma_{rr} = P_i \sim 0!$

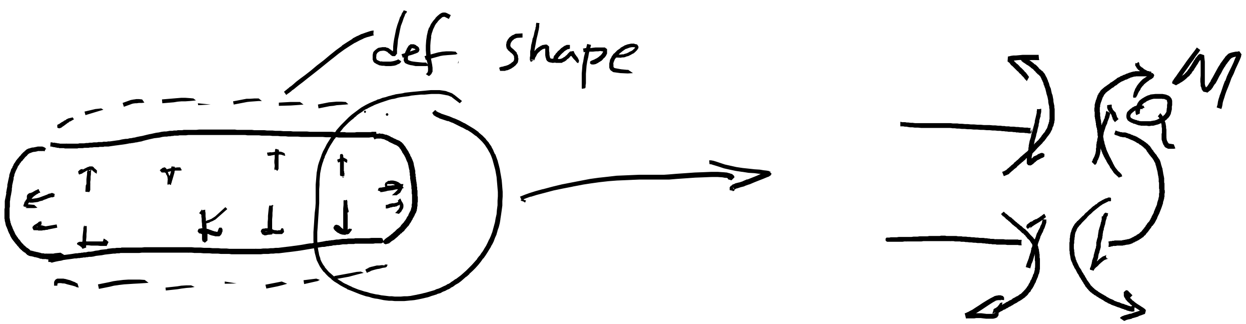
$\boxed{\alpha \sim 55^\circ}$

$\sigma_{zz} = \frac{1}{2}\,\sigma_{\varphi\varphi}$

Laminate theory

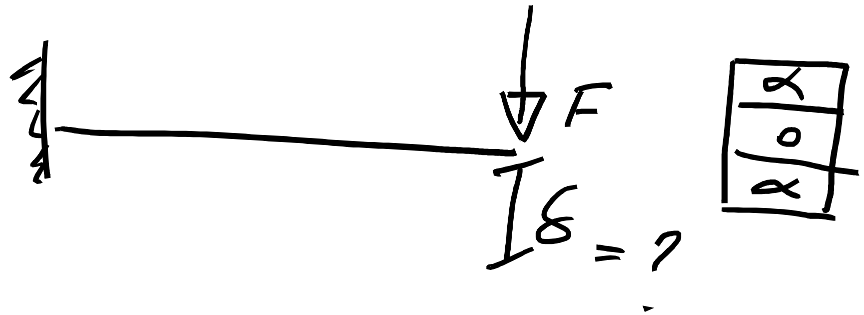
$$\begin{pmatrix} N_x = \sigma_{\varphi\varphi} \cdot t \\ N_y = \sigma_{zz} \cdot t \\ N_{xy} = 0 \end{pmatrix} = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

Stresses at the end Cap



additional stresses!
could lead to delamination!

Example ② Cantilever beam made of FRP



Laminate theory

$D_{11} = ?$
$D_{22} = ?$ $\Rightarrow E_{xb} = ?$

$$\delta = f(F, l, E_{xb}) = \frac{Fl^3}{3E_{xb}I}$$

< Laminate Theory >

List of symbols :

u, v, w : Laminate displacements in x, y, z directions

$u^\circ, v^\circ, w^\circ$: Laminate mid-plane displacements

$\varepsilon_x, \varepsilon_y, \varepsilon_z, \kappa_x, \kappa_y, \kappa_z$: strains and curvatures in global laminate directions

$\varepsilon_x^\circ, \varepsilon_y^\circ, \varepsilon_z^\circ, \kappa_x^\circ, \kappa_y^\circ, \kappa_z^\circ$: mid-plain strains and curvatures

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$: laminate shear strains in global directions

$\begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix}^K$: Lamina stiffness matrix in global direction (Lamina No. k)

N_x (or n_x) ; N_y (or n_y) ; N_{xy} (or n_{xy}) ; M_x (or m_x) ; - - - : resultant forces and moments acting in global directions of a laminate

$\left[\begin{array}{cc|cc} A_{11} & A_{12} \cdots & B_{11} & \cdots \\ \vdots & & \vdots & \\ \hline B_{11} & \cdots & D_{11} & \cdots \end{array}\right]$: Laminate global stiffness matrix

A_{ij} : extensional stiffnesses

B_{ij} : Coupling stiffnesses

D_{ij} : bending stiffnesses

α_K ; h_k : Lamina angle, lamina thickness

A^*_{ij} , B^*_{ij} , D^*_{ij} : Laminate compliances (inverse of stiffness matrix)

$[T]^x = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2nm \\ -nm & nm & m^2-n^2 \end{bmatrix}$: Rotation matrix

With $m = \cos \alpha_K$; $n = \sin \alpha_K$

$\hat{E}_x, \hat{E}_y \cdots \hat{G}_{xy}, \cdots \hat{\nu}_{xy}$: Laminate engineering constants